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Super congruences involving Bernoulli polynomials

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Let $p>3$ be a prime, and let a be a rational p-adic integer. Let $\{B_n\}$ and $\{B_n(x)\}$ denote the Bernoulli numbers and Bernoulli polynomials given by $B_0=1$, $\sum_{k=0}^{n-1}\binom nkB_k=0\ (n\ge 2)$ and $B_n(x)=\sum_{k=0}^n\binom nkB_kx^{n-k}\ (n\ge 0)$. In this paper we show that

$$\begin{aligned}&\sum_{k=0}^{p-1}\binom ak\binom{-1-a}k\frac 1{2k-1}\\&\quad\equiv -(2a+1)(2t+1)-p^2t(t+1)(4+(2a+1)B_{p-2}(-a))\pmod{p^3},\\&\sum_{k=0}^{p-1}\binom ak\binom{-1-a}k\frac{2a+1}{2k+1}\equiv 1+2t+p^2t(t+1)B_{p-2}(-a)\pmod{p^3},\end{aligned}$$

where $t=(a-\langle a\rangle_p)/p$ and $\langle a\rangle_p\in\{0,1,\ldots,p-1\}$ is given by $a\equiv\langle a\rangle_p\pmod p$.

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1. Introduction

The Bernoulli numbers $\{B_n\}$ and Bernoulli polynomials $\{B_n(x)\}$ are defined by

$$B_0=1,\ \sum_{k=0}^{n-1}\binom nkB_k=0\ (n\ge 2)\quad\text{and}\quad B_n(x)=\sum_{k=0}^n\binom nkB_kx^{n-k}\ (n\ge 0).$$

The Euler numbers $\{E_n\}$ are defined by $E_0=1$ and $E_n=-\sum_{k=1}^{[n/2]}\binom n{2k}E_{n-2k}\ (n\ge 1)$, where $[a]$ is the greatest integer not exceeding a. In [S5] the author introduced the sequence $\{U_n\}$ given by $U_0=1$ and $U_n=-2\sum_{k=1}^{[n/2]}\binom n{2k}U_{n-2k}\ (n\ge 1)$. It is well known that $B_{2n+1}=0$ and

$E_{2n-1} = U_{2n-1} = 0$ for any positive integer n. $\{B_n\}$, $\{E_n\}$ and $\{U_n\}$ are important sequences and they have many interesting properties and applications. See [B1], [MOS] and [S1-S6].

As pointed out in [S7], we have

$$\binom{-\frac12}{k}^2 = \frac{1}{16^k}\binom{2k}{k}^2,\quad \binom{-\frac13}{k}\binom{-\frac23}{k} = \frac{1}{27^k}\binom{2k}{k}\binom{3k}{k},$$
$$\binom{-\frac14}{k}\binom{-\frac34}{k} = \frac{1}{64^k}\binom{2k}{k}\binom{4k}{2k},\quad \binom{-\frac16}{k}\binom{-\frac56}{k} = \frac{1}{432^k}\binom{3k}{k}\binom{6k}{3k}. \tag{1.1}$$

Let $p > 3$ be a prime. In 2003, based on his work concerning hypergeometric functions and Calabi-Yau manifolds, Rodriguez-Villegas [RV] conjectured the following congruences:

$$\sum_{k=0}^{p-1}\frac{1}{16^k}\binom{2k}{k}^2 \equiv \Big(\frac{-1}{p}\Big) \pmod{p^2},\quad \sum_{k=0}^{p-1}\frac{1}{27^k}\binom{2k}{k}\binom{3k}{k} \equiv \Big(\frac{-3}{p}\Big) \pmod{p^2}, \tag{1.2}$$

$$\sum_{k=0}^{p-1}\frac{1}{64^k}\binom{2k}{k}\binom{4k}{2k} \equiv \Big(\frac{-2}{p}\Big) \pmod{p^2},\quad \sum_{k=0}^{p-1}\frac{1}{432^k}\binom{3k}{k}\binom{6k}{3k} \equiv \Big(\frac{-1}{p}\Big) \pmod{p^2}, \tag{1.3}$$

where $(\frac{\cdot}{p})$ is the Legendre symbol. These congruences were later confirmed by Mortenson [M1-M2].

Let $\mathbb{Z}$ be the set of integers. For a prime p let $\mathbb{Z}_p$ denote the set of rational $p-adic$ integers. For a $p-adic$ integer a let $\langle a\rangle_p \in \{0,1,\ldots,p-1\}$ be given by $a \equiv \langle a\rangle_p \pmod p$. In [S7] the author generalized (1.2) and (1.3) by showing that for any odd prime p and $a \in \mathbb{Z}_p$,

$$\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k} \equiv (-1)^{\langle a\rangle_p} \pmod{p^2}. \tag{1.4}$$

Inspired by (1.4), in [T] Tauraso obtained a congruence for $\sum_{k=1}^{p-1}\frac1k\binom{a}{k}\binom{-1-a}{k} \pmod{p^2}$.

Let $p > 3$ be a prime. By doing calculations with Mathematica, the author's brother Z.W. Sun conjectured that (see [Su, Conjecture 5.12]):

$$\sum_{k=0}^{p-1}\frac{1}{27^k(2k+1)}\binom{2k}{k}\binom{3k}{k} \equiv \Big(\frac{p}{3}\Big) - \frac23 p^2 B_{p-2}\Big(\frac13\Big) \pmod{p^3}, \tag{1.5}$$

$$\sum_{k=0}^{p-1}\frac{1}{64^k(2k+1)}\binom{2k}{k}\binom{4k}{2k} \equiv \Big(\frac{-1}{p}\Big) - 3p^2E_{p-3} \pmod{p^3}, \tag{1.6}$$

$$\sum_{k=0}^{p-1}\frac{1}{432^k(2k+1)}\binom{6k}{3k}\binom{3k}{k} \equiv \Big(\frac{p}{3}\Big) \pmod{p^2}. \tag{1.7}$$

This entices the author to study the sum $\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\frac{1}{2k+1}$.

Suppose $0 < k < 1$, $(x)_0 = 1$ and $(x)_n = x(x+1)\cdots(x+n-1)$ for $n = 1,2,3,\ldots$. The elliptic integral of the first kind is given by

$$K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}.$$

It is known that (see [B2, p.318])

$$K(k)=\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{(1/2)_n^2}{n!^2}k^{2n}=\frac{\pi}{2}\sum_{n=0}^{\infty}\binom{-1/2}{n}^2k^{2n}.$$

The elliptic integral of the second kind is given by

$$E(k)=\int_0^1\sqrt{\frac{1-k^2t^2}{1-t^2}}dt=\int_0^{\frac{\pi}{2}}\sqrt{1-k^2\sin^2\theta}d\theta.$$

It is known that (see [B2, p.318])

$$E(k)=\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{(-\frac12)_n(\frac12)_n}{n!^2}k^{2n}=\frac{\pi}{2}\sum_{n=0}^{\infty}\binom{\frac12}{n}\binom{-\frac12}{n}k^{2n}=-\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{1}{2n-1}\binom{-\frac12}{n}^2k^{2n}.$$

Let $p>3$ be a prime, $a\in\mathbb{Z}_p$ and $t=(a-\langle a\rangle_p)/p$. Inspired by (1.4) and power series for elliptic integrals, we also investigate the sum $\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\frac{1}{2k-1}$. Actually, in this paper we prove that

$$\begin{aligned}\sum_{k=0}^{p-1}&\binom{a}{k}\binom{-1-a}{k}\frac{1}{2k-1}\\&\equiv-(2a+1)(2t+1)-p^2t(t+1)(4+(2a+1)B_{p-2}(-a))\pmod{p^3}\end{aligned}\tag{1.8}$$

and

$$\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\frac{2a+1}{2k+1}\equiv1+2t+p^2t(t+1)B_{p-2}(-a)\pmod{p^3}.\tag{1.9}$$

Taking $a=-\frac13,-\frac14,-\frac16$ in (1.9) and then applying (1.1) we deduce (1.5), (1.6) and (1.7), respectively.

2. The congruence for $\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\frac{1}{2k+1}\pmod{p^3}$

For any positive integer n and variables a and b with $b\notin\{-1,-\frac12,\dots,-\frac1n\}$ let

$$S_n(a,b)=\sum_{k=0}^{n}\binom{a}{k}\binom{-1-a}{k}\frac{1}{bk+1}.\tag{2.1}$$

Then

$$\begin{aligned}&(ab+1)S_n(a,b)-(ab-1)S_n(a-1,b)\\&=\sum_{k=0}^{n}\binom{a}{k}\binom{-1-a}{k}\frac{ab+1}{bk+1}-\sum_{k=0}^{n}\binom{a-1}{k}\binom{-a}{k}\frac{ab-1}{bk+1}\end{aligned}$$

$$= \sum_{k=0}^{n} \binom{a}{k}\binom{-a}{k}\Big(\frac{ab+1}{bk+1}\cdot\frac{a+k}{a} - \frac{ab-1}{bk+1}\cdot\frac{a-k}{a}\Big) = 2\sum_{k=0}^{n}\binom{a}{k}\binom{-a}{k}.$$

By [S7, (4.5)] or induction on n, $\sum_{k=0}^{n}\binom{a}{k}\binom{-a}{k} = \binom{n+a}{n}\binom{n-a}{n} = \binom{a-1}{n}\binom{-a-1}{n}$. Thus,

$$(ab+1)S_n(a,b) - (ab-1)S_n(a-1,b) = 2\binom{a-1}{n}\binom{-a-1}{n}. \tag{2.2}$$

For any positive integer m define $H_m = 1 + \frac{1}{2} + \cdots + \frac{1}{m}$. We also define $H_0 = 0$.

Lemma 2.1 ([S7, Lemma 4.2]). *Let p be an odd prime, $m \in \{1, 2, \dots, p-1\}$ and $t \in \mathbb{Z}_p$. Then*

$$\binom{m+pt-1}{p-1} \equiv \frac{pt}{m} - \frac{p^2t^2}{m^2} + \frac{p^2t}{m}H_m \pmod{p^3}.$$

Lemma 2.2. *Let p be an odd prime, $a \in \mathbb{Z}_p$, $a \not\equiv 0 \pmod p$ and $t = (a - \langle a\rangle_p)/p$. Then*

$$\binom{a-1}{p-1}\binom{-a-1}{p-1} \equiv p^2\frac{t(t+1)}{\langle a\rangle_p^2} + p^3t(t+1)\Big(-\frac{1+2t}{a^3} + 2\frac{H_{\langle a\rangle_p}}{a^2}\Big) \pmod{p^4}.$$

Proof. By Lemma 2.1,

$$\begin{aligned}\binom{a-1}{p-1} = \binom{\langle a\rangle_p + pt - 1}{p-1} &\equiv \frac{pt}{\langle a\rangle_p} + p^2t\Big(-\frac{t}{\langle a\rangle_p^2} + \frac{1}{\langle a\rangle_p}H_{\langle a\rangle_p}\Big)\\ &\equiv \frac{pt}{\langle a\rangle_p} + p^2t\Big(-\frac{t}{a^2} + \frac{H_{\langle a\rangle_p}}{a}\Big) \pmod{p^3}.\end{aligned}$$

From [S7, p.312] we know that $H_{p-1-\langle a\rangle_p} \equiv H_{\langle a\rangle_p} \pmod p$. Thus, from Lemma 2.1 we deduce that

$$\begin{aligned}\binom{-a-1}{p-1} &= \binom{p-\langle a\rangle_p - p(t+1) - 1}{p-1}\\ &\equiv \frac{p(-t-1)}{p-\langle a\rangle_p} + p^2(-t-1)\Big(-\frac{-t-1}{(p-\langle a\rangle_p)^2} + \frac{H_{p-\langle a\rangle_p}}{p-\langle a\rangle_p}\Big)\\ &\equiv \frac{p(t+1)(\langle a\rangle_p + p)}{\langle a\rangle_p^2} - p^2(t+1)\Big(\frac{t+1}{\langle a\rangle_p^2} - \frac{H_{p-\langle a\rangle_p}}{\langle a\rangle_p}\Big)\\ &\equiv \frac{p(t+1)}{\langle a\rangle_p} + p^2(t+1)\Big\{\frac{1}{\langle a\rangle_p^2} - \frac{t+1}{\langle a\rangle_p^2} + \frac{-\frac{1}{\langle a\rangle_p} + H_{p-1-\langle a\rangle_p}}{\langle a\rangle_p}\Big\}\\ &\equiv \frac{p(t+1)}{\langle a\rangle_p} + p^2(t+1)\Big\{-\frac{1+t}{\langle a\rangle_p^2} + \frac{H_{\langle a\rangle_p}}{\langle a\rangle_p}\Big\}\\ &\equiv \frac{p(t+1)}{\langle a\rangle_p} + p^2(t+1)\Big(-\frac{1+t}{a^2} + \frac{H_{\langle a\rangle_p}}{a}\Big) \pmod{p^3}.\end{aligned}$$

Hence,

$$\binom{a-1}{p-1}\binom{-a-1}{p-1}$$

$$\equiv \Big(\frac{pt}{\langle a\rangle_p}+p^2t\Big(-\frac{t}{a^2}+\frac{H_{\langle a\rangle_p}}{a}\Big)\Big)\Big(\frac{p(t+1)}{\langle a\rangle_p}+p^2(t+1)\Big(-\frac{1+t}{a^2}+\frac{H_{\langle a\rangle_p}}{a}\Big)\Big)$$
$$\equiv p^2\frac{t(t+1)}{\langle a\rangle_p^2}+p^3t(t+1)\Big(-\frac{1+2t}{a^3}+2\frac{H_{\langle a\rangle_p}}{a^2}\Big)\pmod{p^4}.$$

This proves the lemma.

For any positive integer n and variable a let

$$T_n(a)=(2a+1)S_n(a,2)=\sum_{k=0}^{n}\binom{a}{k}\binom{-1-a}{k}\frac{2a+1}{2k+1}. \tag{2.3}$$

Lemma 2.3. *Let $p>3$ be a prime and $t\in\mathbb{Z}_p$. Then*

$$T_{p-1}(pt)\equiv 1+2t \pmod{p^3}.$$

Proof. Clearly

$$\begin{aligned}T_{p-1}(pt)&=\sum_{k=0}^{p-1}\binom{pt}{k}(-1)^k\binom{pt+k}{k}\frac{2pt+1}{2k+1}\\&=2pt+1+\sum_{k=1}^{p-1}\frac{(-1)^kpt(pt+k)(p^2t^2-(k-1)^2)\cdots(p^2t^2-1^2)}{k!^2}\cdot\frac{2pt+1}{2k+1}\\&\equiv 2pt+1+\sum_{\substack{k=1\\k\neq\frac{p-1}{2}}}^{p-1}pt(pt+k)\frac{(-1)^k(-1^2)(-2^2)\cdots(-(k-1)^2)}{k!^2}\cdot\frac{2pt+1}{2k+1}\\&\quad+(-1)^{\frac{p-1}{2}}\frac{(2pt+1)t}{pt-\frac{p-1}{2}}\cdot\frac{(p^2t^2-(\frac{p-1}{2})^2)\cdots(p^2t^2-1^2)}{(\frac{p-1}{2}!)^2}\\&\equiv 2pt+1-pt(2pt+1)\sum_{\substack{k=1\\k\neq\frac{p-1}{2}}}^{p-1}\frac{pt+k}{k^2(2k+1)}+\frac{2t(2pt+1)}{2pt+1-p}\Big(1-p^2t^2\sum_{k=1}^{\frac{p-1}{2}}\frac{1}{k^2}\Big)\pmod{p^3}.\end{aligned}$$

From [L] or [S2] we know that $\sum_{k=1}^{p-1}\frac{1}{k^2}\equiv 0\pmod p$ and $\sum_{k=1}^{p-1}\frac1k\equiv 0\pmod{p^2}$. As $\frac{1}{k^2(2k+1)}=\frac{1}{k^2}-\frac{2}{k}+\frac{4}{2k+1}$, we see that

$$\begin{aligned}\sum_{\substack{k=1\\k\neq\frac{p-1}{2}}}^{p-1}\frac{pt+k}{k^2(2k+1)}&=\sum_{\substack{k=1\\k\neq\frac{p-1}{2}}}^{p-1}(pt+k)\Big(\frac{1}{k^2}-\frac{2}{k}+\frac{4}{2k+1}\Big)\\&=\sum_{k=1}^{p-1}(pt+k)\Big(\frac{1}{k^2}-\frac{2}{k}\Big)-\Big(pt+\frac{p-1}{2}\Big)\Big(\frac{1}{(\frac{p-1}{2})^2}-\frac{2}{\frac{p-1}{2}}\Big)+2\sum_{\substack{k=1\\k\neq\frac{p-1}{2}}}^{p-1}\frac{2pt-1+2k+1}{2k+1}\pmod{p^2}\end{aligned}$$

and so

$$\sum_{\substack{k=1\\k\neq\frac{p-1}{2}}}^{p-1}\frac{pt+k}{k^2(2k+1)}\equiv pt\sum_{k=1}^{p-1}\Big(\frac{1}{k^2}-2\frac{1}{k}\Big)+\sum_{k=1}^{p-1}\Big(\frac{1}{k}-2\Big)-pt\Big(\frac{1}{\frac14}-\frac{2}{-\frac12}\Big)-\Big(\frac{1}{\frac{p-1}{2}}-2\Big)$$

$$+2\sum_{\substack{k=1\\k\neq\frac{p-1}{2}}}^{p-1}1+2(2pt-1)\sum_{\substack{k=1\\k\neq\frac{p-1}{2}}}^{p-1}\frac{1}{2k+1}$$

$$\equiv -8pt+2(p+1)+2(2pt-1)\sum_{\substack{k=1\\k\neq\frac{p-1}{2}}}^{p-1}\frac{1}{2k+1}\pmod{p^2}.$$

Using the known congruences $\sum_{k=1}^{p-1}\frac{1}{k}\equiv 0\pmod{p^2}$ (see [L]) and $\sum_{k=1}^{\frac{p-1}{2}}\frac{1}{k^2}\equiv 0\pmod{p}$ (see [S2, Corollary 5.2]) we find that

$$\sum_{\substack{k=1\\k\neq\frac{p-1}{2}}}^{p-1}\frac{1}{2k+1}=\sum_{k=1}^{\frac{p-3}{2}}\frac{1}{2k+1}+\sum_{k=1}^{\frac{p-1}{2}}\frac{1}{2(\frac{p-1}{2}+k)+1}$$

$$=\sum_{k=1}^{\frac{p-3}{2}}\frac{1}{2k+1}+\sum_{k=1}^{\frac{p-1}{2}}\frac{p-2k}{p^2-4k^2}$$

$$\equiv\sum_{k=1}^{\frac{p-3}{2}}\frac{1}{2k+1}+\sum_{k=1}^{\frac{p-1}{2}}\frac{1}{2k}-\frac{p}{4}\sum_{k=1}^{\frac{p-1}{2}}\frac{1}{k^2}$$

$$=\sum_{k=1}^{p-1}\frac{1}{k}-1-\frac{p}{4}\sum_{k=1}^{\frac{p-1}{2}}\frac{1}{k^2}$$

$$\equiv -1\pmod{p^2}.$$

Therefore,

$$\sum_{\substack{k=1\\k\neq\frac{p-1}{2}}}^{p-1}\frac{pt+k}{k^2(2k+1)}\equiv -8pt+2(p+1)-2(2pt-1)=2p(1-6t)+4\pmod{p^2}.$$

By [S2, Corollary 5.2], $\sum_{k=1}^{(p-1)/2}\frac{1}{k^2}\equiv 0\pmod{p}$. Thus, from all the above we deduce that

$$T_{p-1}(pt)\equiv 2pt+1-pt(2pt+1)\sum_{\substack{k=1\\k\neq\frac{p-1}{2}}}^{p-1}\frac{pt+k}{k^2(2k+1)}+\frac{2t(2pt+1)}{2pt+1-p}$$

$$\equiv 2pt+1-pt(2pt+1)(2p(1-6t)+4)+2t\Big(1+\frac{p}{1+(2t-1)p}\Big)$$

$$\equiv 2pt+1-pt(8pt+2p(1-6t)+4)+2t+2tp(1-(2t-1)p)$$

$$=1+2t\pmod{p^3}.$$

This proves the lemma.

Theorem 2.1. *Let $p>3$ be a prime, $a\in\mathbb{Z}_p$ and $t=(a-\langle a\rangle_p)/p$. Then*

$$\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\frac{2a+1}{2k+1}\equiv 1+2t+p^2t(t+1)B_{p-2}(-a)\pmod{p^3}.$$

Moreover, if $a\not\equiv 0\pmod p$, then

$$\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\frac{2a+1}{2k+1}\equiv 1+2t+p^2t(t+1)\Big(\frac{2}{a^2}-B_{p-2}(a)\Big)\pmod{p^3}.$$

Proof. For $a\not\equiv -\frac12\pmod p$ we see that

$$\begin{aligned}\binom{a}{\frac{p-1}{2}}\binom{-1-a}{\frac{p-1}{2}}&=(-1)^{\frac{p-1}{2}}\binom{a}{\frac{p-1}{2}}\binom{a+\frac{p-1}{2}}{\frac{p-1}{2}}\\&=(-1)^{\frac{p-1}{2}}\frac{(a+\frac{p-1}{2})(a+\frac{p-1}{2}-1)\cdots(a-\frac{p-1}{2}+1)}{(\frac{p-1}{2}!)^2}\equiv 0\pmod p.\end{aligned}$$

Thus, $\binom{a}{k}\binom{-1-a}{k}\frac{2a+1}{2k+1}\in\mathbb{Z}_p$ for $k=0,1,\ldots,p-1$. When $a=pt\equiv 0\pmod p$, by [S2, Lemma 3.1] we have $B_{p-2}(-a)=B_{p-2}(-pt)-B_{p-2}(0)\equiv 0\pmod p$. Thus, the first result follows from Lemma 2.3. Now suppose that $a\not\equiv 0\pmod p$ and $T_n(a)$ is given by (2.3). By (2.2) and Lemma 2.2,

$$\begin{aligned}&T_{p-1}(a)-T_{p-1}(a-1)\\&=2\binom{a-1}{p-1}\binom{-a-1}{p-1}\equiv 2p^2\frac{t(t+1)}{\langle a\rangle_p^2}+2p^3t(t+1)\Big(-\frac{1+2t}{a^3}+2\frac{H_{\langle a\rangle_p}}{a^2}\Big)\pmod{p^4}.\end{aligned}$$

For $1\le k\le\langle a\rangle_p$ we have $\langle a-k+1\rangle_p=\langle a\rangle_p-k+1$ and so $a-k+1=\langle a\rangle_p-k+1+pt=\langle a-k+1\rangle_p+pt$. Hence

$$\begin{aligned}&T_{p-1}(a)-T_{p-1}(a-\langle a\rangle_p)\\&=\sum_{k=1}^{\langle a\rangle_p}(T_{p-1}(a-k+1)-T_{p-1}(a-k))\equiv\sum_{k=1}^{\langle a\rangle_p}\frac{2t(t+1)p^2}{\langle a-k+1\rangle_p^2}\\&=2t(t+1)p^2\sum_{k=1}^{\langle a\rangle_p}\frac{1}{(\langle a\rangle_p-k+1)^2}=2t(t+1)p^2\sum_{r=1}^{\langle a\rangle_p}\frac{1}{r^2}\equiv 2t(t+1)p^2\sum_{r=1}^{\langle a\rangle_p}r^{p-3}\pmod{p^3}.\end{aligned}$$

By [S2, Lemma 3.2],

$$\sum_{r=1}^{\langle a\rangle_p}r^{p-3}\equiv(-1)^{p-2}\frac{B_{p-2}(-a)-B_{p-2}}{p-2}\equiv\frac12 B_{p-2}(-a)\pmod p.$$

Thus,

$$T_{p-1}(a)-T_{p-1}(pt)$$

$$= T_{p-1}(a) - T_{p-1}(a - \langle a\rangle_p) \equiv 2t(t+1)p^2 \cdot \frac{1}{2}B_{p-2}(-a) = p^2 t(t+1)B_{p-2}(-a) \pmod{p^3}.$$

This together with Lemma 2.3 yields $T_{p-1}(a) \equiv 1 + 2t + p^2 t(t+1)B_{p-2}(-a) \pmod{p^3}$. From [MOS] we know that $B_n(-a) = (-1)^n(B_n(a) + na^{n-1})$. Thus,

$$B_{p-2}(-a) = (-1)^{p-2}(B_{p-2}(a) + (p-2)a^{p-3}) \equiv -B_{p-2}(a) + \frac{2}{a^2} \pmod{p}.$$

This completes the proof.

Corollary 2.1. *Let $p > 3$ be a prime and $a \in \mathbb{Z}_p$ with $a \not\equiv 0 \pmod{p}$. Then*

$$\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\frac{2a+1}{2k+1} + \sum_{k=0}^{p-1}\binom{-a}{k}\binom{-1+a}{k}\frac{1-2a}{2k+1} \equiv (a - \langle a\rangle_p)(p + a - \langle a\rangle_p)\frac{2}{a^2} \pmod{p^3}.$$

Proof. As $\langle -a\rangle_p = p - \langle a\rangle_p$, from Theorem 2.1 we derive that

$$\begin{aligned}
&\sum_{k=0}^{p-1}\binom{-a}{k}\binom{-1+a}{k}\frac{1-2a}{2k+1} \\
&\equiv 1 + 2\frac{-a - \langle -a\rangle_p}{p} + (-a - \langle -a\rangle_p)(p - a - \langle -a\rangle_p)\Big(\frac{2}{a^2} - B_{p-2}(-a)\Big) \\
&= -1 - 2\frac{a - \langle a\rangle_p}{p} + (a - \langle a\rangle_p)(p + a - \langle a\rangle_p)\Big(\frac{2}{a^2} - B_{p-2}(-a)\Big) \\
&\equiv (a - \langle a\rangle_p)(p + a - \langle a\rangle_p)\frac{2}{a^2} - \sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\frac{2a+1}{2k+1} \pmod{p^3}.
\end{aligned}$$

This yields the result.

For $m = 3, 4, 6$ it is clear that

$$-\frac{1}{m} - \Big\langle -\frac{1}{m}\Big\rangle_p = \begin{cases} -\frac{1}{m} - \frac{p-1}{m} = -\frac{p}{m} & \text{if } p \equiv 1 \pmod{m}, \\ -\frac{1}{m} - \frac{(m-1)p-1}{m} = -\frac{(m-1)p}{m} & \text{if } p \equiv -1 \pmod{m} \end{cases} \tag{2.4}$$

and so

$$\Big(-\frac{1}{m} - \Big\langle -\frac{1}{m}\Big\rangle_p\Big)\Big(p - \frac{1}{m} - \Big\langle -\frac{1}{m}\Big\rangle_p\Big) = -\frac{p}{m}\cdot\frac{(m-1)p}{m} = -\frac{m-1}{m^2}p^2. \tag{2.5}$$

Theorem 2.2. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{1}{64^k(2k+1)}\binom{2k}{k}\binom{4k}{2k} \equiv (-1)^{\frac{p-1}{2}} - 3p^2 E_{p-3} \pmod{p^3}.$$

Proof. Taking $a = -\frac{1}{4}$ in Theorem 2.1 and then applying (1.1), (2.4) and (2.5) we obtain

$$\sum_{k=0}^{p-1}\frac{1}{64^k(2k+1)}\binom{2k}{k}\binom{4k}{2k} \equiv \frac{1 - 2\cdot\frac{2-(\frac{-1}{p})}{4}}{1 - 2\cdot\frac{1}{4}} + p^2\frac{-\frac{1}{4}\cdot\frac{3}{4}}{1 - 2\cdot\frac{1}{4}}B_{p-2}\Big(\frac{1}{4}\Big)$$

$$= (-1)^{\frac{p-1}{2}} - \frac{3}{8}p^2 B_{p-2}\Big(\frac{1}{4}\Big) \pmod{p^3}.$$

It is known (see for example [S4, Lemma 2.5]) that $E_{2n} = -4^{2n+1}\frac{B_{2n+1}(1/4)}{2n+1}$. Thus, $E_{p-3} = -4^{p-2}\frac{B_{p-2}(1/4)}{p-2} \equiv \frac{B_{p-2}(1/4)}{8} \pmod p$. Now combining all the above we obtain the result.

Theorem 2.3. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{1}{27^k(2k+1)}\binom{2k}{k}\binom{3k}{k} \equiv \Big(\frac{p}{3}\Big) - 4p^2U_{p-3} \pmod{p^3}.$$

Proof. Taking $a = -\frac{1}{3}$ in Theorem 2.1 and then applying (1.1), (2.4) and (2.5) we obtain (1.5). By [S5, p.217], $B_{p-2}(\frac{1}{3}) \equiv 6U_{p-3} \pmod p$. Thus the result follows.

Theorem 2.4. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{1}{432^k(2k+1)}\binom{6k}{3k}\binom{3k}{k} \equiv \Big(\frac{p}{3}\Big) - \frac{25}{4}p^2U_{p-3} \pmod{p^3}.$$

Proof. Taking $a = -\frac{1}{6}$ in Theorem 2.1 and then applying (1.1), (2.4) and (2.5) we obtain

$$\sum_{k=0}^{p-1}\frac{1}{432^k(2k+1)}\binom{6k}{3k}\binom{3k}{k} \equiv \Big(\frac{p}{3}\Big) - \frac{5}{24}p^2B_{p-2}\Big(\frac{1}{6}\Big) \pmod{p^3}.$$

By [S5, p.216], $B_{p-2}(\frac{1}{6}) \equiv 30U_{p-3} \pmod p$. Thus the result follows.

3. The congruence for $\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\frac{1}{2k-1} \pmod{p^3}$

Lemma 3.1. *For any nonnegative integer n we have*

$$\sum_{k=0}^{n}\binom{a}{k}\binom{-1-a}{k}\frac{(2a(a+1)+1)k - a(a+1)}{4k^2-1} = \frac{a(a+1)}{2n+1}\binom{a-1}{n}\binom{-2-a}{n}.$$

Proof. It is easy to check that

$$\begin{aligned}&\frac{a(a+1)}{2(n+1)+1}\binom{a-1}{n+1}\binom{-2-a}{n+1} - \frac{a(a+1)}{2n+1}\binom{a-1}{n}\binom{-2-a}{n}\\ &=\binom{a}{n+1}\binom{-1-a}{n+1}\Big\{\frac{a(a+1)}{2n+3}\cdot\frac{a-n-1}{a}\cdot\frac{-2-a-n}{-1-a} - \frac{a(a+1)}{2n+1}\cdot\frac{n+1}{a}\cdot\frac{n+1}{-1-a}\Big\}\\ &=\binom{a}{n+1}\binom{-1-a}{n+1}\frac{(2a(a+1)+1)(n+1) - a(a+1)}{4(n+1)^2-1}.\end{aligned}$$

Thus the result can be easily proved by induction on n.

Lemma 3.2. *Let p be an odd prime and $a\in\mathbb{Z}_p$. Then*

$$\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\frac{(2a(a+1)+1)k-a(a+1)}{4k^2-1}\equiv -(a-\langle a\rangle_p)(p+a-\langle a\rangle_p)\pmod{p^3}.$$

Proof. Set $a=\langle a\rangle_p+pt$. From Lemma 2.1 we see that

$$\binom{a-1}{p-1}=\binom{\langle a\rangle_p+pt-1}{p-1}\equiv\frac{pt}{\langle a\rangle_p}\equiv\frac{pt}{a}\pmod{p^2}\quad\text{for}\quad a\not\equiv 0\pmod p,$$
$$\binom{-2-a}{p-1}=\binom{p-1-\langle a\rangle_p-p(t+1)-1}{p-1}\equiv\frac{-p(t+1)}{p-1-\langle a\rangle_p}\equiv\frac{p(t+1)}{a+1}\pmod{p^2}$$
$$\text{for}\quad a\not\equiv -1\pmod p.$$

Thus,

$$\begin{aligned}&\frac{a(a+1)}{2(p-1)+1}\binom{a-1}{p-1}\binom{-2-a}{p-1}\\&\equiv\begin{cases}\frac{a(a+1)}{2p-1}\cdot\frac{pt}{a}\cdot\frac{p(t+1)}{a+1}\equiv -p^2t(t+1)\pmod{p^3} & \text{if } a\not\equiv 0,-1\pmod p,\\ \frac{pt(pt+1)}{2p-1}\binom{pt-1}{p-1}\frac{p(t+1)}{pt+1}\equiv -p^2t(t+1)\pmod{p^3} & \text{if } a\equiv 0\pmod p,\\ \frac{(p-1+pt)(p+pt)}{2p-1}\cdot\frac{pt}{p-1+pt}\binom{p-1-p(t+2)}{p-1}\equiv -p^2t(t+1)\pmod{p^3} & \\ & \text{if } a\equiv -1\pmod p.\end{cases}\end{aligned}$$

Now taking $n=p-1$ in Lemma 3.1 and then applying the above we obtain the result.

Theorem 3.1. *Let $p>3$ be a prime, $a\in\mathbb{Z}_p$ and $t=(a-\langle a\rangle_p)/p$. Then*

$$\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\frac{1}{2k-1}\equiv -(2a+1)(2t+1)-p^2t(t+1)(4+(2a+1)B_{p-2}(-a))\pmod{p^3}.$$

Proof. Note that $\frac{1}{2k-1}=4\frac{(2a(a+1)+1)k-a(a+1)}{4k^2-1}-\frac{(2a+1)^2}{2k+1}$. Combining Theorem 2.1 with Lemma 3.2 we deduce the result.

Corollary 3.1. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{1}{64^k(2k-1)}\binom{2k}{k}\binom{4k}{2k}\equiv -\frac14\Big(\frac{-1}{p}\Big)+\frac34p^2(1+E_{p-3})\pmod{p^3}.$$

Proof. Taking $a=-\frac14$ in Theorem 3.1 and then applying (1.1), (2.4), (2.5) and the aforementioned fact that $B_{p-2}(\frac14)\equiv 8E_{p-3}\pmod p$ we deduce the result.

Corollary 3.2. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{1}{27^k(2k-1)}\binom{2k}{k}\binom{3k}{k}\equiv -\frac19\Big(\frac{-3}{p}\Big)+\frac49p^2(2+U_{p-3})\pmod{p^3}$$

and

$$\sum_{k=0}^{p-1}\frac{1}{432^k(2k-1)}\binom{6k}{3k}\binom{3k}{k}\equiv -\frac{4}{9}\Big(\frac{-3}{p}\Big)+\frac{5}{9}p^2(1+5U_{p-3}) \pmod{p^3}.$$

Proof. Taking $a=-\frac13,-\frac16$ in Theorem 3.1 and then applying (1.1), (2.4), (2.5) and the facts (see [S5, pp.216-217]) that $B_{p-2}(\frac13)\equiv 6U_{p-3} \pmod p$ and $B_{p-2}(\frac16)\equiv 30U_{p-3} \pmod p$ we deduce the above two congruences, respectively.

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