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Supercongruences involving products of three binomial coefficients via Beukers' method

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Abstract. Recently, using modular forms F. Beukers posed a unified method that can deal with a large number of supercongruences involving binomial coefficients and Apéry-like numbers. In this paper, we use Beukers' method to prove some conjectures of the first author concerning the congruences for

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{m^k},\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{m^k}\quad\text{and}\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m^k}$$

modulo p^3, where p is an odd prime representable by some suitable binary quadratic form and m is an integer not divisible by p.

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1. Introduction

In 1998, Ono[13] established congruences for $\sum_{k=0}^{p-1}\binom{2k}{k}^3\frac{1}{m^k}$ modulo p in the cases $m=1,-8,16,-64,256,-512,4096$ for any prime $p\neq 2,7$. Following these, for such values of m, in [22] the first author's brother Z.W. Sun conjectured congruences for $\sum_{k=0}^{p-1}\binom{2k}{k}^3\frac{1}{m^k}$ modulo p^2, which have been proved by the first author in [16] and Kibelbek et al in [9]. Going beyond that, in [19], the first author also conjecturally formulated congruences for $\sum_{k=0}^{p-1}\binom{2k}{k}^3\frac{1}{m^k}$ modulo p^3. Before instantiating these congruences, hereafter, for positive integers a, b and n, if $n = ax^2 + by^2$ for some integers x and y, we simply write that $n = ax^2 + by^2$, and recall the well known results (see [6,21]) that for an odd prime p,

$$\begin{aligned}
p &= x^2 + 4y^2 \quad \text{for} \quad p \equiv 1 \pmod 4,\\
p &= x^2 + 3y^2 \quad \text{for} \quad p \equiv 1 \pmod 3,\\
p &= x^2 + 2y^2 \quad \text{for} \quad p \equiv 1, 3 \pmod 8,\\
p &= x^2 + 7y^2 \quad \text{for} \quad p \equiv 1, 2, 4 \pmod 7,
\end{aligned}$$

where the integers x and y are uniquely determined up to sign. Following these, an example of the first author's conjectures can be stated as that for any prime $p \neq 2, 7$,

$$\sum_{k=0}^{p-1}\binom{2k}{k}^3 \equiv \begin{cases} 4x^2 - 2p - \frac{p^2}{4x^2} \pmod{p^3} \\ \qquad\qquad \text{if } p \equiv 1,2,4 \pmod 7 \text{ and so } p = x^2 + 7y^2, \\ -11p^2\binom{[3p/7]}{[p/7]}^{-2} \pmod{p^3} & \text{if } p \equiv 3 \pmod 7, \\ -\frac{11}{16}p^2\binom{[3p/7]}{[p/7]}^{-2} \pmod{p^3} & \text{if } p \equiv 5 \pmod 7, \\ -\frac{11}{4}p^2\binom{[3p/7]}{[p/7]}^{-2} \pmod{p^3} & \text{if } p \equiv 6 \pmod 7, \end{cases} \tag{1.1}$$

where $[a]$ is the greatest integer not exceeding a.

Based on the work of Long and Ramakrishna[8], the first author[20] illustrated that for any odd prime p,

$$\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^3}{64^k} \equiv \begin{cases} 4x^2 - 2p - \frac{p^2}{4x^2} \pmod{p^3} & \text{if } p = x^2 + 4y^2 \equiv 1 \pmod 4, \\ -p^2\binom{\frac{p-1}{2}}{\frac{p-3}{4}}^{-2} \pmod{p^3} & \text{if } p \equiv 3 \pmod 4 \end{cases} \tag{1.2}$$

and

$$(-1)^{\frac{p-1}{4}}\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^3}{(-512)^k} \equiv 4x^2 - 2p - \frac{p^2}{4x^2} \pmod{p^3} \text{ for } p = 4n + 1 = x^2 + 4y^2. \tag{1.3}$$

What's more, in [19] and [20], the first author posed many conjectural congruences for

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{m^k}, \quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{m^k} \quad \text{and} \quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m^k}$$

modulo p^3 for prime $p>3$ and integer m coprime to p. See also [15-18] and [24] for relevant congruences modulo p^2. For example, the first author[19] experimentally found that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k}\equiv\begin{cases}4x^2-2p-\frac{p^2}{4x^2}\pmod{p^3} & \text{if } p=x^2+2y^2\equiv 1,3\pmod 8,\\ \frac{p^2}{3}\binom{[p/4]}{[p/8]}^{-2}\pmod{p^3} & \text{if } p\equiv 5\pmod 8,\\ -\frac{3}{2}p^2\binom{[p/4]}{[p/8]}^{-2}\pmod{p^3} & \text{if } p\equiv 7\pmod 8,\end{cases}\tag{1.4}$$

$$\left(\frac{-3}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}}\equiv\begin{cases}4x^2-2p-\frac{p^2}{4x^2}\pmod{p^3} & \text{if } p=x^2+4y^2\equiv 1\pmod 4,\\ \frac{5}{12}p^2\binom{(p-3)/2}{(p-3)/4}^{-2}\pmod{p^3} & \text{if } p\equiv 3\pmod 4,\end{cases}\tag{1.5}$$

where $\left(\frac{a}{p}\right)$ is the Jacobi symbol. It is worth remarking that Liu[7] formulated the supercongruences for the sums in (1.4)-(1.5) modulo p^3 in terms of p-adic Gamma functions. In [10] , using p-adic Gamma functions and Jacobi sums Mao proved (1.4) in the cases $p\equiv 1,5,7\pmod 8$. The congruences (1.4) and (1.5) modulo p^2 were conjectured by Rodriguez-Villegas[14] and proved by Mortenson[12] and Z.W. Sun[23].

Recently, in order to conceptually interpret all these conjectures, using modular forms Beukers[2] found a unified way to deal with supercongruences modulo p^2 or p^3 for sums involving binomial coefficients and Apéry-like numbers, where p is an odd prime representable by some suitable binary quadratic form. In this paper, using Beukers' method we prove a number of congruences conjectured by the first author. In particular, we rigorously verify a number of CM values of modular functions that were stated without proofs in Beukers' work [2].

Thanks to Beukers for his summary [2, Appendix C] of products of three binomial coefficients, Apéry-like numbers and their associated Hauptmoduls, as well as findings of their related CM values, we notice that a number of conjectures of the first author can also be charted similarly. In what follows, we state all the supercongruences involving products of three binomial coefficients that we have been able to attain based on Beukers' work.

The following congruences that we shall prove in Sections 3 and 4 were conjectured by the first author in [19] and [20].

Theorem 1.1 *Let p be an odd prime, $p\equiv 1,2,4\pmod 7$ and so $p=x^2+7y^2$.*

Then

$$\sum_{k=0}^{(p-1)/2}\binom{2k}{k}^3 \equiv (-1)^{\frac{p-1}{2}}\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{4096^k} \equiv 4x^2-2p-\frac{p^2}{4x^2} \pmod{p^3}, \tag{1.6}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{81^k} \equiv \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-3969)^k} \equiv 4x^2-2p-\frac{p^2}{4x^2} \pmod{p^3}. \tag{1.7}$$

Theorem 1.2 *Let p be a prime of the form $3k+1$ and so $p=x^2+3y^2$. Then*

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{16^k} \equiv (-1)^{\frac{p-1}{2}}\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{256^k} \equiv 4x^2-2p-\frac{p^2}{4x^2} \pmod{p^3}, \tag{1.8}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-144)^k} \equiv 4x^2-2p-\frac{p^2}{4x^2} \pmod{p^3}. \tag{1.9}$$

Theorem 1.3 *Let p be a prime of the form $4k+1$ and so $p=x^2+4y^2$. Then*

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{(-8)^k} \equiv 4x^2-2p-\frac{p^2}{4x^2} \pmod{p^3}, \tag{1.10}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{648^k} \equiv 4x^2-2p-\frac{p^2}{4x^2} \pmod{p^3}. \tag{1.11}$$

Theorem 1.4 *Let p be a prime such that $p\equiv 1,3 \pmod 8$ and so $p=x^2+2y^2$. Then*

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{(-64)^k} \equiv (-1)^{\frac{p-1}{2}}\left(4x^2-2p-\frac{p^2}{4x^2}\right) \pmod{p^3}, \tag{1.12}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k} \equiv 4x^2-2p-\frac{p^2}{4x^2} \pmod{p^3}, \tag{1.13}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{28^{4k}} \equiv 4x^2-2p-\frac{p^2}{4x^2} \pmod{p^3}. \tag{1.14}$$

Theorem 1.5 *Let p be a prime of the form $4k+1$. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-12288)^k} \equiv \begin{cases} 4x^2-2p-\frac{p^2}{4x^2} \pmod{p^3} & \text{if } p\equiv 1 \pmod{12} \text{ and so } p=x^2+9y^2, \\ -2x^2+2p+\frac{p^2}{2x^2} \pmod{p^3} & \text{if } p\equiv 5 \pmod{12} \text{ and so } 2p=x^2+9y^2 \end{cases}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-6635520)^k} \equiv \begin{cases} 4x^2-2p-\frac{p^2}{4x^2} \pmod{p^3} & \text{if } p\equiv 1,9 \pmod{20} \text{ and so } p=x^2+25y^2, \\ -2x^2+2p+\frac{p^2}{2x^2} \pmod{p^3} & \text{if } p\equiv 13,17 \pmod{20} \text{ and so } 2p=x^2+25y^2. \end{cases}$$

Theorem 1.6 *Let $m\in\{5,13,37\}$ and $D(m)=-1024,-82944,-14112^2$ according as $m=5,13,37$. Suppose that p is an odd prime such that $(\frac{-m}{p})=1$. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{D(m)^k} \equiv \begin{cases} 4x^2-2p-\frac{p^2}{4x^2} \pmod{p^3} \\ \qquad \text{if } (\frac{-1}{p})=(\frac{m}{p})=1 \text{ and so } p=x^2+my^2, \\ -2x^2+2p+\frac{p^2}{2x^2} \pmod{p^3} \\ \qquad \text{if } (\frac{-1}{p})=(\frac{m}{p})=-1 \text{ and so } 2p=x^2+my^2. \end{cases}$$

Theorem 1.7 *Let $m\in\{3,5,11,29\}$ and $F(m)=48^2,12^4,1584^2,396^4$ according as $m=3,5,11,29$. Suppose that p is an odd prime such that $(\frac{-2m}{p})=1$. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{F(m)^k} \equiv \begin{cases} 4x^2-2p-\frac{p^2}{4x^2} \pmod{p^3} \\ \qquad \text{if } (\frac{-(-1)^{\frac{m-1}{2}}2}{p})=(\frac{(-1)^{\frac{m-1}{2}}m}{p})=1 \text{ and so } p=x^2+2my^2, \\ -8x^2+2p+\frac{p^2}{8x^2} \pmod{p^3} \\ \qquad \text{if } (\frac{-(-1)^{\frac{m-1}{2}}2}{p})=(\frac{(-1)^{\frac{m-1}{2}}m}{p})=-1 \text{ and so } p=2x^2+my^2. \end{cases}$$

Here the representability of p and $2p$ by x^2+my^2, x^2+2my^2 or $2x^2+my^2$ is guaranteed by [21, Table 9.1].

It is noteworthy that the case of $p \equiv 1 \pmod 8$ of (1.12)-(1.13) have been treated by Mao[10], and congruences in (1.9) and (1.11) in the weaker form under modulo p^2 were proved by Wang and Z.W. Sun[25], and the congruence for $\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{4k}{2k}/81^k$ modulo p^2 and the congruences modulo p^2 for the sums in (1.14) and Theorems 1.5-1.7 were first conjecturally formulated by Z.W. Sun[22,24]. In addition, the case $m = 5$ in Theorem 1.6 has been treated by Beukers in [2, Corollary 1.19], while his result was stated in different form, and he did not give a proof of the corresponding rational CM value.

Combining Theorem 1.27 in Beukers' work [2] and the work [18] of the first author, in Section 5 we also confirm a number of conjectures of the first author given in [19] and [20] that can be stated as follows.

Theorem 1.8 *Let p be a prime of the form $4k+1$ and so $p = x^2 + 4y^2$. Then*

$$\left(\frac{-3}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}} \equiv \left(\frac{33}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{66^{3k}} \equiv 4x^2 - 2p - \frac{p^2}{4x^2} \pmod{p^3}.$$

Theorem 1.9 *Let p be a prime of the form $3k+1$ and so $p = x^2 + 3y^2$. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{54000^k} \equiv \left(\frac{5}{p}\right)\left(4x^2 - 2p - \frac{p^2}{4x^2}\right) \pmod{p^3}.$$

Theorem 1.10 *Let p be a prime such that $p \equiv 1, 3 \pmod 8$ and so $p = x^2 + 2y^2$. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{20^{3k}} \equiv \left(\frac{-5}{p}\right)\left(4x^2 - 2p - \frac{p^2}{4x^2}\right) \pmod{p^3}.$$

Theorem 1.11 *Let p be an odd prime, $p \equiv 1, 2, 4 \pmod 7$ and so $p = x^2 + 7y^2$. Then*

$$\left(\frac{-15}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-15)^{3k}} \equiv \left(\frac{-255}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{255^{3k}} \equiv 4x^2 - 2p - \frac{p^2}{4x^2} \pmod{p^3}.$$

Theorem 1.12 *Let p be a prime of the form $3k+1$ and so $4p = x^2 + 27y^2$. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-12288000)^k} \equiv \left(\frac{10}{p}\right)\left(x^2 - 2p - \frac{p^2}{x^2}\right) \pmod{p^3}.$$

Theorem 1.13 *Let $m \in \{11, 19, 43, 67, 163\}$ and p be a prime such that $(\frac{p}{m}) = 1$ and so $4p = x^2 + my^2$, and let*

$$m_1 = \begin{cases} -32 & \text{if } m = 11, \\ -96 & \text{if } m = 19, \\ -960 & \text{if } m = 43, \\ -5280 & \text{if } m = 67, \\ -640320 & \text{if } m = 163 \end{cases} \quad \text{and} \quad m_0 = \begin{cases} -2 & \text{if } m = 11, \\ -6 & \text{if } m = 19, \\ -15 & \text{if } m = 43, \\ -330 & \text{if } m = 67, \\ -10005 & \text{if } m = 163. \end{cases}$$

Then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m_1^{3k}} \equiv \left(\frac{m_0}{p}\right)\left(x^2 - 2p - \frac{p^2}{x^2}\right) \pmod{p^3}.$$

It is noteworthy to remark that the congruences for the sums modulo p^2 in Theorem 1.13 were conjectured earlier by Z.W. Sun[24].

2. Preliminaries

Let $\mathbb{R}$ be the set of real numbers and $H = \{a + bi \mid a, b \in \mathbb{R},\ b > 0\}$. Let

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \; a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\},$$

and for any positive integer N,

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \pmod{N} \right\}.$$

For $\tau \in H$ the Dedekind eta function $\eta(\tau)$ is defined by

$$\eta(\tau) = e^{2\pi i\tau/24} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i\tau n}\right).$$

It is well known (see [6, Corollary 12.19]) that

$$\eta(\tau + 1) = e^{2\pi i/24}\eta(\tau) \quad \text{and} \quad \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau).$$

For $\tau \in H$ let

$$\Delta(\tau) = \eta(\tau)^{24} = q\prod_{n=1}^{\infty}(1-q^n)^{24} \quad \text{with} \quad q = e^{2\pi i\tau}.$$

Then

$$\Delta\Big(-\frac{1}{\tau}\Big) = \eta\Big(-\frac{1}{\tau}\Big)^{24} = (\sqrt{-i\tau})^{24}\eta(\tau)^{24} = \tau^{12}\Delta(\tau). \tag{2.1}$$

Moreover, it is well known (see [3]) that for $\tau \in H$,

$$\Delta\Big(\frac{a\tau+b}{c\tau+d}\Big) = (c\tau+d)^{12}\Delta(\tau) \quad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}). \tag{2.2}$$

For $\tau \in H$ let

$$g_2(\tau) = 60 \sum_{m,n\in\mathbb{Z},(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^4}.$$

The modular function $j(\tau)$ is defined by

$$j(\tau) = 1728\frac{g_2(\tau)^3}{(2\pi)^{12}\eta(\tau)^{24}}.$$

It is well known (see [3] and [6, (12.4) and (12.6)]) that

$$\begin{aligned} &j(\tau+1) = j(\tau), \quad j\Big(-\frac{1}{\tau}\Big) = j(\tau), \\ &j\Big(\frac{a\tau+b}{c\tau+d}\Big) = j(\tau) \quad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}). \end{aligned} \tag{2.3}$$

For $c, d \in \mathbb{Z}$ let (c, d) be the greatest common divisor of c and d. For given imaginary quadratic irrational number α let $\bar{\alpha}$ be the conjugate of α. For an odd prime p let $\mathbb{Z}_p$ be the ring of p-adic integers. Let $X_0(N)^+$ be the modular curve associated to the Fricke group $\Gamma_0(N)^+$, the Fuchsian group generated by $\Gamma_0(N)$ and its involution $\begin{pmatrix} 0 & 1 \\ -N & 0 \end{pmatrix}$. In particular, recall that for a Fuchsian group Γ commensurable with $\mathrm{SL}_2(\mathbb{Z})$ of genus zero, a Hauptmodul for Γ is a modular function for Γ with a unique pole.

In [2], Beukers established the following great theorem.

Beukers' theorem ([2, Theorems 1.15-1.16]) *Let* $F(t) = \sum_{n=0}^{\infty} a_n t^n$. *Suppose that*

(1) *for* $\tau \in H$, $t(\tau)$ *is a Hauptmodul for the modular group* $\Gamma_0(N)^+$.

(2) $F(t(\tau))$ *is a modular form of weight* 2 *with respect to* $\Gamma_0(N)^+$. *In particular,* $F(t(\tau))$ *satisfies that* $F(t(-1/(N\tau))) = -N\tau^2 F(t(\tau))$.

(3) $F(t(\tau))$ *has a unique zero (modulo the action of* $\Gamma_0(N)^+$*), which is located at the pole of* $t(\tau)$. *Assume that it has order* $1/r$ *for some positive integer* r.

(4) $F(t(\tau))$ *can be written as an* η*-product or* $r \geq 4$.

Let α *be an imaginary quadratic number with positive imaginary part. Let* p *be a prime not dividing* N *which splits in the quadratic field* $\mathbb{Q}(\alpha)$.

(i) *Suppose that there exist integers* c *and* d *such that*

$$\begin{aligned}&(c,d)=1,\ N \mid c, \quad c\alpha\bar{\alpha}, c(\alpha+\bar{\alpha}) \in \mathbb{Z},\\ &p=(c\alpha+d)(c\bar{\alpha}+d) \quad \text{and} \quad c\alpha+d \notin p\mathbb{Z}_p.\end{aligned} \tag{2.4}$$

If $t(\alpha) \in \mathbb{Z}_p$, *then*

$$\sum_{n=0}^{p-1} a_n t(\alpha)^n \equiv (c\alpha+d)^2 \pmod{p^3}.$$

(ii) *Suppose that there exist integers* c *and* d *such that*

$$\begin{aligned}&(c,dN)=1,\ Nc\alpha\bar{\alpha} \in \mathbb{Z}, \quad c(\alpha+\bar{\alpha}) \in \mathbb{Z},\\ &p=N(c\alpha+d)(c\bar{\alpha}+d) \quad \text{and} \quad c\alpha+d \notin p\mathbb{Z}_p.\end{aligned} \tag{2.5}$$

If $t(\alpha) \in \mathbb{Z}_p$, *then*

$$\sum_{n=0}^{p-1} a_n t(\alpha)^n \equiv -N(c\alpha+d)^2 \pmod{p^3}.$$

In order to prove our main results, we need the following basic lemma.

Lemma 2.1 *Let* p *be an odd prime, and let* $d > 1$ *be an integer not divisible by* p. *Suppose that* c *is a positive integer,* $cp = x^2 + dy^2 (x, y \in \mathbb{Z})$ *and* $x + y\sqrt{-d} \notin p\mathbb{Z}_p$. *Then*

$$\begin{aligned}x+y\sqrt{-d} &\equiv 2x - \frac{cp}{2x} - \frac{c^2p^2}{8x^3} - \frac{c^3p^3}{16x^5} \pmod{p^4},\\ (x+y\sqrt{-d})^2 &\equiv 4x^2 - 2cp - \frac{c^2p^2}{4x^2} - \frac{c^3p^3}{8x^4} \pmod{p^4}.\end{aligned}$$

Proof. Suppose $A = x + y\sqrt{-d}$. Since $(A-x)^2 = -dy^2 = x^2 - cp$, we have $A(A-2x) = -cp$ and so $A \equiv 2x \pmod p$. Set $A = 2x + \frac{kp}{2x}$. Then $(2x + \frac{kp}{2x})\frac{kp}{2x} = -cp$ and so $k + \frac{k^2p}{4x^2} = -c$. Hence

$$\begin{aligned} A &= 2x + \frac{kp}{2x} = 2x - \frac{(c+\frac{k^2p}{4x^2})p}{2x} = 2x - \frac{cp}{2x} - \frac{k^2p^2}{8x^3} \\ &= 2x - \frac{cp}{2x} - \frac{p^2}{8x^3}\Big(-c - \frac{k^2p}{4x^2}\Big)^2 \equiv 2x - \frac{cp}{2x} - \frac{p^2}{8x^3}\Big(c^2 + \frac{ck^2p}{2x^2}\Big) \\ &\equiv 2x - \frac{cp}{2x} - \frac{c^2p^2}{8x^3} - \frac{c^3p^3}{16x^5} \pmod{p^4}. \end{aligned}$$

Therefore,

$$\begin{aligned} A^2 &\equiv \Big(2x - \frac{cp}{2x} - \frac{c^2p^2}{8x^3} - \frac{c^3p^3}{16x^5}\Big)^2 \\ &\equiv \Big(2x - \frac{cp}{2x}\Big)^2 - 2\Big(2x - \frac{cp}{2x}\Big)\Big(\frac{c^2p^2}{8x^3} + \frac{c^3p^3}{16x^5}\Big) \\ &\equiv 4x^2 - 2cp + \frac{c^2p^2}{4x^2} - \frac{c^2p^2}{2x^2} + \frac{c^3p^3}{8x^4} - \frac{c^3p^3}{4x^4} \\ &= 4x^2 - 2cp - \frac{c^2p^2}{4x^2} - \frac{c^3p^3}{8x^4} \pmod{p^4}. \end{aligned}$$

This completes the proof.

3. Proofs of congruences involving $\binom{2k}{k}^3$

Lemma 3.1 *For $\tau \in H$ let $t(\tau)$ be defined by*

$$t(\tau) = \frac{q}{\prod_{n=1}^{\infty}(1+q^{2n-1})^{24}} \quad \textit{with} \quad q = e^{2\pi i\tau}.$$

Then

$$t(\tau) = \frac{\Delta(\tau)\Delta(4\tau)}{\Delta(2\tau)^2} = t\Big(-\frac{1}{4\tau}\Big), \quad (16t(\tau)-1)^3 + j(2\tau)t(\tau)^2 = 0$$

and $t(\tau)$ is a Hauptmodul for $\Gamma_0(4)^+$ with a unique pole at the cusp $[\frac{1}{2}]$.

Proof. For $\tau \in H$ and $q = e^{2\pi i\tau}$ we have

$$\prod_{n=1}^{\infty}\frac{1}{1+q^{2n-1}} = \prod_{n=1}^{\infty}\frac{1-q^{2n-1}}{1-q^{4n-2}} = \prod_{n=1}^{\infty}\frac{1-q^n}{1-q^{2n}}\cdot\frac{1-q^{4n}}{1-q^{2n}}.$$

Thus,

$$t(\tau) = q\prod_{n=1}^{\infty}\frac{(1-q^n)^{24}(1-q^{4n})^{24}}{(1-q^{2n})^{48}} = \frac{\Delta(\tau)\Delta(4\tau)}{\Delta(2\tau)^2}.$$

Using (2.1) we see that

$$t\Big(-\frac{1}{4\tau}\Big) = \frac{\Delta(-\frac{1}{4\tau})\Delta(-\frac{1}{\tau})}{\Delta(-\frac{1}{2\tau})^2} = \frac{(4\tau)^{12}\Delta(4\tau)\cdot\tau^{12}\Delta(\tau)}{(2\tau)^{24}\Delta(2\tau)^2} = t(\tau).$$

Appealing to (2.2),

$$t\left(\frac{a\tau+b}{c\tau+d}\right) = \frac{\Delta\big(\frac{a\tau+b}{c\tau+d}\big)\Delta\big(\frac{a(4\tau)+4b}{(c/4)(4\tau)+d}\big)}{\Delta\big(\frac{a(2\tau)+2b}{(c/2)(2\tau)+d}\big)^2} = \frac{\Delta(\tau)\Delta(4\tau)}{\Delta(2\tau)^2} = t(\tau) \text{ for } \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma_0(4).$$

Notice that $t(\tau)$ has no zeros or poles in H by its infinite product representation, and $t(\tau) = q + O(q^2)$ has a simple zero at the cusp $[i\infty]$ by its definition. Since $X_0(4)^+$ has only cusps $[i\infty]$ and $[\frac{1}{2}]$, and $t(\tau)$ is a modular function on $X_0(4)^+$ with no zeros or poles in H, using the Residue Theorem for compact Riemann surfaces one can see that this forces $t(\tau)$ to have a simple pole at the cusp $[\frac{1}{2}]$, which is the unique pole of $t(\tau)$ in $X_0(4)^+$. Hence, $t(\tau)$ is a Hauptmodul for $\Gamma_0(4)^+$.

From [6, Theorem 12.17],

$$j(\tau) = \Big(\frac{f(\tau)^{24}-16}{f(\tau)^8}\Big)^3 \quad \text{for} \quad f(\tau) = e^{-\frac{2\pi i}{48}}\frac{\eta((\tau+1)/2)}{\eta(\tau)}. \tag{3.1}$$

Observe that

$$\begin{aligned} f(2\tau)^{24} &= -\frac{\eta(\tau+\frac{1}{2})^{24}}{\eta(2\tau)^{24}} = -\prod_{n=1}^{\infty}\frac{e^{2\pi i(\tau+\frac{1}{2})}(1-e^{2\pi i(\tau+\frac{1}{2})n})^{24}}{e^{2\pi i\cdot 2\tau}(1-e^{2\pi i\cdot 2\tau n})^{24}} \\ &= \frac{1}{e^{2\pi i\tau}}\prod_{n=1}^{\infty}\frac{(1-(-1)^n e^{2\pi i\tau n})^{24}}{(1-e^{2\pi i\tau\cdot 2n})^{24}} = \frac{1}{e^{2\pi i\tau}}\prod_{n=1}^{\infty}\Big(1+e^{2\pi i\tau(2n-1)}\Big)^{24} = \frac{1}{t(\tau)}. \end{aligned}$$

We derive that

$$j(2\tau) = \frac{(f(2\tau)^{24}-16)^3}{f(2\tau)^{24}} = \frac{(\frac{1}{t(\tau)}-16)^3}{\frac{1}{t(\tau)}} = \frac{(1-16t(\tau))^3}{t(\tau)^2}$$

and so $(16t(\tau)-1)^3 + j(2\tau)t(\tau)^2 = 0$.

Lemma 3.2 *Suppose that b is a positive real number. For $a \in \mathbb{Z}$ we have $(-1)^a t(\frac{a+\sqrt{-b}}{2}) > 0$. For $a \in \mathbb{R}$ we have $\overline{t(a+\sqrt{-b})} = t(-a+\sqrt{-b})$.*

Proof. For $a \in \mathbb{Z}$ we have $e^{2\pi i \cdot \frac{a+\sqrt{-b}}{2}} = (-1)^a e^{-\sqrt{b}\pi}$ and so

$$(-1)^a t\Big(\frac{a+\sqrt{-b}}{2}\Big) = \frac{e^{-\sqrt{b}\pi}}{\prod_{n=1}^{\infty}(1+(-1)^a e^{-\sqrt{b}\pi(2n-1)})^{24}} > 0.$$

For $a \in \mathbb{R}$ we see that $\overline{e^{2\pi i(a+\sqrt{-b})}} = \overline{e^{2\pi i a - 2\pi\sqrt{b}}} = e^{-2\pi i a - 2\pi\sqrt{b}} = e^{2\pi i(-a+\sqrt{-b})}$ and so

$$\begin{aligned}\overline{t(a+\sqrt{-b})} &= \overline{\frac{e^{2\pi i(a+\sqrt{-b})}}{\prod_{n=1}^{\infty}(1+e^{2\pi i(a+\sqrt{-b})(2n-1)})^{24}}} = \frac{e^{2\pi i(-a+\sqrt{-b})}}{\prod_{n=1}^{\infty}(1+e^{2\pi i(-a+\sqrt{-b})(2n-1)})^{24}} \\ &= t(-a+\sqrt{-b}).\end{aligned}$$

Lemma 3.3 *For $\tau \in H$ let $t(\tau)$ be given in Lemma 3.1. Then*

$$\begin{gathered} t\Big(\frac{3+\sqrt{-7}}{8}\Big) = 1, \quad t\Big(\frac{\sqrt{-7}}{2}\Big) = \frac{1}{4096}, \quad t\Big(\frac{3+\sqrt{-3}}{4}\Big) = \frac{1}{16}, \\ t\Big(\frac{\sqrt{-3}}{2}\Big) = \frac{1}{256}, \quad t\Big(\frac{1+i}{2}\Big) = -\frac{1}{8}, \quad t\Big(\frac{1+\sqrt{-2}}{2}\Big) = -\frac{1}{64}. \end{gathered}$$

Proof. From [6, (12.20)] we have $j(\frac{3+\sqrt{-7}}{2}) = -15^3$. Since $j(\tau+1) = j(\tau)$ and $j(\tau) = j(-\frac{1}{\tau})$, we see that

$$\begin{aligned} j\Big(\frac{3+\sqrt{-7}}{4}\Big) &= j\Big(\frac{-1+\sqrt{-7}}{4}\Big) = j\Big(-\frac{4}{-1+\sqrt{-7}}\Big) \\ &= j\Big(\frac{1+\sqrt{-7}}{2}\Big) = j\Big(\frac{3+\sqrt{-7}}{2}\Big) = -15^3. \end{aligned}$$

Hence, appealing to Lemma 3.1, $t(\frac{3+\sqrt{-7}}{8})$ is a root of $(16x-1)^3 - 15^3x^2 = (x-1)(4096x^2-47x+1) = 0$ and so $t\big(\frac{3+\sqrt{-7}}{8}\big) \in \big\{1, \frac{47+45\sqrt{-7}}{8192}, \frac{47-45\sqrt{-7}}{8192}\big\}$. By Lemmas 3.1 and 3.2,

$$\overline{t\Big(\frac{3+\sqrt{-7}}{8}\Big)} = t\Big(\frac{-3+\sqrt{-7}}{8}\Big) = t\Big(-\frac{1}{4\cdot\frac{-3+\sqrt{-7}}{8}}\Big) = t\Big(\frac{3+\sqrt{-7}}{8}\Big).$$

Hence, $t\big(\frac{3+\sqrt{-7}}{8}\big)$ is real and therefore $t(\frac{3+\sqrt{-7}}{8}) = 1$.

From [6, (12.20)], $j(\sqrt{-7}) = 255^3$. Hence $t(\frac{\sqrt{-7}}{2})$ is a root of $(16x-1)^3+255^3x^2 = 0$ by Lemma 3.1. Observe that $(16x-1)^3+255^3x^2 = (4096x-1)(x^2+4048x+1)$. We have $t\big(\frac{\sqrt{-7}}{2}\big) \in \big\{\frac{1}{4096}, -2024+\sqrt{2024^2-1}, -2024-\sqrt{2024^2-1}\big\}$. By Lemma 3.2, $t(\frac{\sqrt{-7}}{2}) > 0$. Thus, $t\big(\frac{\sqrt{-7}}{2}\big) = \frac{1}{4096}$.

Since $j(\tau+1) = j(\tau)$, from [6, p.261] we have $j\big(\frac{3+\sqrt{-3}}{2}\big) = j\big(\frac{1+\sqrt{-3}}{2}\big) = 0$ and so $t\big(\frac{3+\sqrt{-3}}{4}\big) = \frac{1}{16}$ by Lemma 3.1.

From [6, pp.261,291] we have $j(\sqrt{-3}) = 54000$. Thus, applying Lemma 3.1 we see that $t(\frac{\sqrt{-3}}{2})$ is a root of $(16x-1)^3+54000x^2 = (256x-1)(16x^2+208x+1) = 0$ and so $t\big(\frac{\sqrt{-3}}{2}\big) \in \big\{\frac{1}{256}, \frac{-26+15\sqrt{3}}{4}, \frac{-26-15\sqrt{3}}{4}\big\}$. By Lemma 3.2, $t(\frac{\sqrt{-3}}{2}) > 0$. Thus $t\big(\frac{\sqrt{-3}}{2}\big) = \frac{1}{256}$.

By [6, (12.20)], $j(i) = 12^3$ and so $j(1+i) = 12^3$ since $j(\tau+1) = j(\tau)$. From Lemma 3.1, $t(\frac{1+i}{2})$ is a root of $(16x-1)^3+12^3x^2 = (8x+1)^2(64x-1) = 0$. Thus, $t(\frac{1+i}{2}) = -\frac{1}{8}$ or $\frac{1}{64}$. From Lemma 3.2, $t(\frac{1+i}{2}) < 0$. Thus, $t(\frac{1+i}{2}) = -\frac{1}{8}$.

By [6, (12.20)], $j(\sqrt{-2}) = 20^3$. Thus, $j(1+\sqrt{-2}) = j(\sqrt{-2}) = 20^3$. Hence $t(\frac{1+\sqrt{-2}}{2})$ is a root of $(16x-1)^3+20^3x^2 = 0$ by Lemma 3.1. Since $(16x-1)^3+20^3x^2 = (64x+1)(64x^2+112x-1)$, we see that $t(\frac{1+\sqrt{-2}}{2}) \in \{-\frac{1}{64}, \frac{-7+5\sqrt{2}}{8}, \frac{-7-5\sqrt{2}}{8}\}$. By Lemma 3.2, $t(\frac{1+\sqrt{-2}}{2}) < 0$ and so $t(\frac{1+\sqrt{-2}}{2}) \in \{-\frac{1}{64}, \frac{-7-5\sqrt{2}}{8}\}$. Note that $e^{2\pi i(1+\sqrt{-2})/2} = -e^{-\sqrt{2}\pi}$. By Bernoulli's inequality $(1+a_1)(1+a_2)\cdots(1+a_n) > 1+a_1+a_2+\cdots+a_n$ for $a_1, a_2, \ldots, a_n \in (-1,0)$, we have

$$\prod_{n=1}^{\infty}\big(1-e^{-\sqrt{2}\pi(2n-1)}\big) \ge 1-\sum_{n=1}^{\infty} e^{-\sqrt{2}\pi(2n-1)} = 1-\frac{e^{-\sqrt{2}\pi}}{1-e^{-2\sqrt{2}\pi}} > \frac{1}{e^{\frac{\sqrt{2}\pi}{24}}}.$$

Therefore,

$$t\Big(\frac{1+\sqrt{-2}}{2}\Big) = -\frac{1}{e^{\sqrt{2}\pi}\prod_{n=1}^{\infty}\big(1-e^{-\sqrt{2}\pi(2n-1)}\big)^{24}} > -1$$

and so $t(\frac{1+\sqrt{-2}}{2}) = -\frac{1}{64}$. This completes the proof.

Remark 3.1 Lemma 3.3 was stated by Beukers in [2, p.29] without proof.

Lemma 3.4 *Let $t(\tau)$ be given in Lemma 3.1. For $\tau \in H$ we have*

$$\sum_{n=0}^{\infty}\binom{2n}{n}^3 t(\tau)^n = \frac{\eta(2\tau)^{20}}{\eta(\tau)^8\eta(4\tau)^8}$$

is a weight 2 modular form for $\Gamma_0(4)^+$ with a unique zero at the cusp $[\frac{1}{2}]$ of order 1.

Proof. By Lemma 3.1, $t(\tau) = \left(\frac{\eta(\tau)\eta(4\tau)}{\eta(2\tau)^2}\right)^{24}$. Thus, the equality in Lemma 3.4 was already proved by Cooper in [5, Theorem 4.1(d)]. From [11], for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$,

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \left(\frac{a}{c_0}\right)\zeta_{24}^{ab+cd(1-a^2)-ca+3c_0(a-1)+r\frac{3}{2}(a^2-1)}(c\tau+d)^{\frac{1}{2}}\eta(\tau), \tag{3.2}$$

where $\zeta_n = e^{2\pi i/n}$ and c_0 is given by $c = 2^r c_0$ with $2 \nmid c_0$. Set $h(\tau) = \frac{\eta(2\tau)^{20}}{\eta(\tau)^8\eta(4\tau)^8}$. Using (3.2) we see that

$$\begin{aligned} h\Big(\frac{a\tau+b}{c\tau+d}\Big) &= \frac{\eta(\frac{a(2\tau)+2b}{(c/2)(2\tau)+d})^{20}}{\eta(\frac{a\tau+b}{c\tau+d})^8\eta(\frac{a(4\tau)+4b}{(c/4)(4\tau)+d})^8} \\ &= \frac{\zeta_{24}^{20(a(2b)+(c/2)d(1-a^2)-\frac{1}{2}ca+3c_0(a-1)+(r-1)\frac{3}{2}(a^2-1))}((c/2)(2\tau)+d)^{10}\eta(2\tau)^{20}}{\zeta_{24}^{8(ab+cd(1-a^2)-ca+3c_0(a-1)+r\frac{3}{2}(a^2-1))}(c\tau+d)^4\eta(\tau)^8} \\ &\quad\times \frac{1}{\zeta_{24}^{8(a(4b)+(c/4)d(1-a^2)-\frac{1}{4}ca+3c_0(a-1)+(r-2)\frac{3}{2}(a^2-1))}((c/4)(4\tau)+d)^4\eta(4\tau)^8} \\ &= (c\tau+d)^2\frac{\eta(2\tau)^{20}}{\eta(\tau)^8\eta(4\tau)^8} = (c\tau+d)^2 h(\tau) \quad \text{for any } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4). \end{aligned}$$

Since $\eta(-\frac{1}{\tau}) = \sqrt{-i\tau}\,\eta(\tau)$ we see that $h(-\frac{1}{4\tau}) = \frac{\eta(-1/(2\tau))^{20}}{\eta(-1/\tau)^8\eta(-1/(4\tau))^8} = -4\tau^2 h(\tau)$. Therefore, $h(\tau)$ is a weight 2 modular form for $\Gamma_0(4)^+$. Moreover, by its infinite product representation, it is clear that $h(\tau)$ has no zeros or poles in H, and $h(\tau) = 1 + O(q)$ is of order 0 at the cusp $[i\infty]$, where $q = e^{2\pi i\tau}$. So since $X_0(4)^+$ has only two cusps $[i\infty]$ and $[\frac{1}{2}]$, and $h(\tau)$ is a weight 2 modular form for $\Gamma_0(4)^+$ with no zeros or poles apart from the cusp $[\frac{1}{2}]$, then by the Riemann-Roch theorem this forces $h(\tau)$ to have a zero of order 1 at the cusp $[\frac{1}{2}]$. The proof is now complete.

Let $t(\tau)$ be given in Lemma 3.1. From Lemmas 3.1 and 3.4 one can see that both $t(\tau)$ and $\sum_{n=0}^{\infty}\binom{2n}{n}^3 t(\tau)^n$ satisfy the assumptions in Beukers' theorem.

Proof of (1.6). Since $p = x^2 + 7y^2 = (x + y\sqrt{-7})(x - y\sqrt{-7})$ for $x, y \in \mathbb{Z}$, one may choose the sign of y so that $x + y\sqrt{-7} \notin p\mathbb{Z}_p$. Set $a_n = \binom{2n}{n}^3$, $c = 8y$, $d = x - 3y$, $\alpha = \frac{3+\sqrt{-7}}{8}$ and $N = 4$. Then clearly $(c, d) = 1$, $N \mid c$, $c\alpha\bar{\alpha}, c(\alpha + \bar{\alpha}) \in \mathbb{Z}$, $p = (c\alpha + d)(c\bar{\alpha} + d)$ and $c\alpha + d \notin p\mathbb{Z}_p$. By Lemma 3.3, $t(\frac{3+\sqrt{-7}}{8}) = 1$. Hence, applying Beukers' theorem(i) and Lemma 2.1 we obtain

$$\sum_{n=0}^{p-1}\binom{2n}{n}^3=\sum_{n=0}^{p-1}\binom{2n}{n}^3 t\Big(\frac{3+\sqrt{-7}}{8}\Big)^n\equiv\Big(8y\cdot\frac{3+\sqrt{-7}}{8}+x-3y\Big)^2$$
$$=(x+y\sqrt{-7})^2\equiv 4x^2-2p-\frac{p^2}{4x^2}\pmod{p^3}.$$

For $p=x^2+7y^2\equiv 1\pmod 4$ we have $2\mid y$. Set $a_n=\binom{2n}{n}^3,\ c=2y,\ d=x,\ \alpha=\frac{\sqrt{-7}}{2}$ and $N=4$. Then $(c,d)=1,\ N\mid c,\quad c\alpha\bar{\alpha},c(\alpha+\bar{\alpha})\in\mathbb{Z},\ p=(c\alpha+d)(c\bar{\alpha}+d)$ and $c\alpha+d\notin p\mathbb{Z}_p$. Since $t\big(\frac{\sqrt{-7}}{2}\big)=\frac{1}{4096}$ by Lemma 3.3, using Beukers' theorem(i) and Lemma 2.1 we see that

$$\sum_{n=0}^{p-1}\frac{\binom{2n}{n}^3}{4096^n}=\sum_{n=0}^{p-1}\binom{2n}{n}^3 t\Big(\frac{\sqrt{-7}}{2}\Big)^n\equiv\Big(2y\cdot\frac{\sqrt{-7}}{2}+x\Big)^2\equiv 4x^2-2p-\frac{p^2}{4x^2}\pmod{p^3}.$$

For $p=x^2+7y^2\equiv 3\pmod 4$ we have $2\mid x$ and $2\nmid y$. Set $a_n=\binom{2n}{n}^3,\ c=y,\ d=\frac{x}{2},\ \alpha=\frac{\sqrt{-7}}{2}$ and $N=4$. Then $(c,dN)=1,\ Nc\alpha\bar{\alpha}\in\mathbb{Z},\quad c(\alpha+\bar{\alpha})\in\mathbb{Z},\ p=N(c\alpha+d)(c\bar{\alpha}+d)$ and $c\alpha+d\notin p\mathbb{Z}_p$. Since $t\big(\frac{\sqrt{-7}}{2}\big)=\frac{1}{4096}$ by Lemma 3.3, using Beukers' theorem(ii) and Lemma 2.1 we see that

$$\sum_{n=0}^{p-1}\frac{\binom{2n}{n}^3}{4096^n}=\sum_{n=0}^{p-1}\binom{2n}{n}^3 t\Big(\frac{\sqrt{-7}}{2}\Big)^n\equiv -4\Big(y\cdot\frac{\sqrt{-7}}{2}+\frac{x}{2}\Big)^2$$
$$=-(x+y\sqrt{-7})^2\equiv-\Big(4x^2-2p-\frac{p^2}{4x^2}\Big)\pmod{p^3}.$$

Note that $p\mid\binom{2k}{k}$ and so $p^3\mid\binom{2k}{k}^3$ for $k=\frac{p+1}{2},\ldots,p-1$. Summarizing the above proves (1.6).

Proof of (1.8). Since $p=x^2+3y^2=(x+y\sqrt{-3})(x-y\sqrt{-3})$ for $x,y\in\mathbb{Z}$, one may choose the sign of y so that $x+y\sqrt{-3}\notin p\mathbb{Z}_p$. Set $a_n=\binom{2n}{n}^3,\ c=4y,\ d=x-3y,\ \alpha=\frac{3+\sqrt{-3}}{4}$ and $N=4$. Then clearly (2.4) holds. By Lemma 3.3, $t(\frac{3+\sqrt{-3}}{4})=\frac{1}{16}$. Hence, applying Beukers' theorem(i) and Lemma 2.1 we obtain

$$\sum_{n=0}^{p-1}\frac{\binom{2n}{n}^3}{16^n}=\sum_{n=0}^{p-1}\binom{2n}{n}^3 t\Big(\frac{3+\sqrt{-3}}{4}\Big)^n\equiv\Big(4y\cdot\frac{3+\sqrt{-3}}{4}+x-3y\Big)^2$$
$$=(x+y\sqrt{-3})^2\equiv 4x^2-2p-\frac{p^2}{4x^2}\pmod{p^3}.$$

For $p=x^2+3y^2\equiv 1 \pmod 4$ we have $2\mid y$. Set $a_n=\binom{2n}{n}^3,\ c=2y,\ d=x,\ \alpha=\frac{\sqrt{-3}}{2}$ and $N=4$. Then (2.4) holds. Since $t\big(\frac{\sqrt{-3}}{2}\big)=\frac{1}{256}$ by Lemma 3.3, using Beukers' theorem(i) and Lemma 2.1 we see that

$$\sum_{n=0}^{p-1}\frac{\binom{2n}{n}^3}{256^n}=\sum_{n=0}^{p-1}\binom{2n}{n}^3 t\Big(\frac{\sqrt{-3}}{2}\Big)^n\equiv\Big(2y\cdot\frac{\sqrt{-3}}{2}+x\Big)^2\equiv 4x^2-2p-\frac{p^2}{4x^2} \pmod{p^3}.$$

For $p=x^2+3y^2\equiv 3 \pmod 4$ we have $2\mid x$ and $2\nmid y$. Set $a_n=\binom{2n}{n}^3,\ c=y,\ d=\frac{x}{2},\ \alpha=\frac{\sqrt{-3}}{2}$ and $N=4$. Then (2.5) holds. Since $t\big(\frac{\sqrt{-3}}{2}\big)=\frac{1}{256}$ by Lemma 3.3, using Beukers' theorem(ii) and Lemma 2.1 we see that

$$\begin{aligned}\sum_{n=0}^{p-1}\frac{\binom{2n}{n}^3}{256^n}&=\sum_{n=0}^{p-1}\binom{2n}{n}^3 t\Big(\frac{\sqrt{-3}}{2}\Big)^n\equiv -4\Big(y\cdot\frac{\sqrt{-3}}{2}+\frac{x}{2}\Big)^2\\&=-(x+y\sqrt{-3})^2\equiv-\Big(4x^2-2p-\frac{p^2}{4x^2}\Big) \pmod{p^3}.\end{aligned}$$

Note that $p\mid\binom{2k}{k}$ and so $p^3\mid\binom{2k}{k}^3$ for $k=\frac{p+1}{2},\ldots,p-1$. From the above we deduce (1.8).

Proof of (1.10). Since $p=(x+2yi)(x-2yi)$, we may choose the sign of x so that $x+2yi\notin p\mathbb{Z}_p$. Set $a_n=\binom{2n}{n}^3,\ c=4y,\ d=x-2y,\ \alpha=\frac{1+i}{2}$ and $N=4$. Then (2.4) holds. By Lemma 3.3, $t(\frac{1+i}{2})=-\frac{1}{8}$. Hence, applying Beukers' theorem(i) and Lemma 2.1 we obtain

$$\begin{aligned}\sum_{n=0}^{p-1}\frac{\binom{2n}{n}^3}{(-8)^n}&=\sum_{n=0}^{p-1}\binom{2n}{n}^3 t\Big(\frac{1+i}{2}\Big)^n\equiv\Big(4y\cdot\frac{1+i}{2}+x-2y\Big)^2\\&=(x+y\sqrt{-4})^2\equiv 4x^2-2p-\frac{p^2}{4x^2} \pmod{p^3}.\end{aligned}$$

Since $p\mid\binom{2k}{k}$ and so $p^3\mid\binom{2k}{k}^3$ for $\frac{p}{2}<k\le p-1$, the result follows.

Proof of (1.12). Since $p=x^2+2y^2=(x+y\sqrt{-2})(x-y\sqrt{-2})$ for $x,y\in\mathbb{Z}$, one may choose the sign of y so that $x+y\sqrt{-2}\notin p\mathbb{Z}_p$.

For $p=x^2+2y^2\equiv 1 \pmod 8$ we have $2\mid y$. Set $a_n=\binom{2n}{n}^3,\ c=2y,\ d=x-y,\ \alpha=\frac{1+\sqrt{-2}}{2}$ and $N=4$. Then (2.4) holds. Since $t\big(\frac{1+\sqrt{-2}}{2}\big)=-\frac{1}{64}$ by Lemma

3.3, using Beukers' theorem(i) and Lemma 2.1 we see that

$$\begin{aligned}\sum_{n=0}^{p-1}\frac{\binom{2n}{n}^3}{(-64)^n}&=\sum_{n=0}^{p-1}\binom{2n}{n}^3 t\Big(\frac{1+\sqrt{-2}}{2}\Big)^n\equiv\Big(2y\cdot\frac{1+\sqrt{-2}}{2}+x-y\Big)^2\\&=(x+y\sqrt{-2})^2\equiv 4x^2-2p-\frac{p^2}{4x^2}\pmod{p^3}.\end{aligned}$$

For $p=x^2+2y^2\equiv 3\pmod 8$ we have $2\nmid xy$. Set $a_n=\binom{2n}{n}^3$, $c=y$, $d=\frac{x-y}{2}$, $\alpha=\frac{1+\sqrt{-2}}{2}$ and $N=4$. Then (2.5) holds. Since $t\big(\frac{1+\sqrt{-2}}{2}\big)=-\frac{1}{64}$ by Lemma 3.3, using Beukers' theorem(ii) and Lemma 2.1 we see that

$$\begin{aligned}\sum_{n=0}^{p-1}\frac{\binom{2n}{n}^3}{(-64)^n}&=\sum_{n=0}^{p-1}\binom{2n}{n}^3 t\Big(\frac{1+\sqrt{-2}}{2}\Big)^n\equiv -4\Big(y\cdot\frac{1+\sqrt{-2}}{2}+\frac{x-y}{2}\Big)^2\\&=-(x+y\sqrt{-2})^2\equiv -\Big(4x^2-2p-\frac{p^2}{4x^2}\Big)\pmod{p^3}.\end{aligned}$$

Note that $p\mid\binom{2k}{k}$ and so $p^3\mid\binom{2k}{k}^3$ for $k=\frac{p+1}{2},\ldots,p-1$. From the above (1.12) is proved.

4. Proofs of congruences involving $\binom{2k}{k}^2\binom{4k}{2k}$

For given rational number n set

$$\sigma(n)=\begin{cases}\sum_{d\mid n}d & \text{if } n\in\{1,2,3,\ldots\},\\ 0 & \text{otherwise.}\end{cases}$$

For $\tau\in H$ define

$$h_2(\tau)=\frac{\Delta(2\tau)}{\Delta(\tau)},\quad u(\tau)=\frac{h_2(\tau)}{(1+64h_2(\tau))^2}\quad\text{and}\quad u_1(\tau)=\frac{1}{u(\tau)}.\tag{4.1}$$

Then

$$h_2(\tau)=\frac{e^{2\pi i\cdot 2\tau}}{e^{2\pi i\tau}}\prod_{n=1}^{\infty}\Big(\frac{1-e^{2\pi i\cdot 2\tau n}}{1-e^{2\pi i\tau n}}\Big)^{24}=\frac{q}{\prod_{n=1}^{\infty}(1-q^{2n-1})^{24}}\quad\text{for}\quad q=e^{2\pi i\tau}.\tag{4.2}$$

By (2.1),

$$h_2\Big(-\frac{1}{2\tau}\Big)=\frac{\Delta(-\frac{1}{\tau})}{\Delta(-\frac{1}{2\tau})}=\frac{\tau^{12}\Delta(\tau)}{(2\tau)^{12}\Delta(2\tau)}=\frac{1}{4096h_2(\tau)}.\tag{4.3}$$

Lemma 4.1 *For $\tau \in H$ we have*

$$u(\tau) = u\Big(-\frac{1}{2\tau}\Big) \quad \text{and} \quad u_1(\tau)^2 - 207u_1(\tau) + 3456 - (j(\tau) + j(2\tau)) = 0.$$

Proof. By (4.3),

$$u\Big(-\frac{1}{2\tau}\Big) = \frac{\frac{1}{4096h_2(\tau)}}{(1+\frac{64}{4096h_2(\tau)})^2} = \frac{h_2(\tau)}{(1+64h_2(\tau))^2} = u(\tau).$$

From [6, Theorem 12.17],

$$j(\tau) = \frac{(4096h_2(\tau)+16)^3}{4096h_2(\tau)} = h_2\Big(\frac{\tau}{2}\Big)\Big(\frac{1}{h_2(\frac{\tau}{2})} + 16\Big)^3. \tag{4.4}$$

Thus,

$$\begin{aligned} j(\tau) + j(2\tau) &= \frac{(256h_2(\tau)+1)^3}{h_2(\tau)} + h_2(\tau)\Big(\frac{1}{h_2(\tau)} + 16\Big)^3 \\ &= 2^{24}h_2(\tau)^2 + \frac{1}{h_2(\tau)^2} + 49\Big(2^{12}h_2(\tau) + \frac{1}{h_2(\tau)}\Big) + 1536 \\ &= \Big(2^{12}h_2(\tau) + \frac{1}{h_2(\tau)}\Big)^2 + 49\Big(2^{12}h_2(\tau) + \frac{1}{h_2(\tau)}\Big) - 6656 \\ &= (u_1(\tau) - 128)^2 + 49(u_1(\tau) - 128) - 6656 \\ &= u_1(\tau)^2 - 207u_1(\tau) + 3456. \end{aligned}$$

This proves the lemma.

Lemma 4.2 *For $\tau \in H$ let $u(\tau)$ be given in (4.1) and $E_2(\tau)$ denote the normalized weight 2 Eisenstein series*

$$E_2(\tau) = 1 - 24\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} = 1 - 24\sum_{n=1}^{\infty} \sigma(n)q^n.$$

Then $u(\tau) = \big(\frac{\eta(\tau)^2\eta(2\tau)^2}{8E_2(2\tau)-4E_2(\tau)}\big)^4$ is a Hauptmodul for $\Gamma_0(2)^+$ with a unique pole at $\frac{1+i}{2}$, and for $\tau \in H$ such that $u(\tau)$ is near 0,

$$\sum_{n=0}^{\infty} \binom{2n}{n}^2 \binom{4n}{2n} u(\tau)^n = 2E_2(2\tau) - E_2(\tau) = 1 + 24\sum_{n=1}^{\infty} \Big(\sigma(n) - 2\sigma\big(\frac{n}{2}\big)\Big)q^n$$

is a weight 2 modular form with respect to $\Gamma_0(2)^+$ with a unique zero of order $\frac{1}{4}$ at $\frac{1+i}{2}$.

Proof. Both identities were proved by Cooper in [4, Theorem 4.30]. For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2)$, using (2.2) we see that

$$h_2\Big(\frac{a\tau+b}{c\tau+d}\Big) = \frac{\Delta\big(\frac{a(2\tau)+2b}{\frac{c}{2}(2\tau)+d}\big)}{\Delta\big(\frac{a\tau+b}{c\tau+d}\big)} = \frac{(c\tau+d)^{12}\Delta(2\tau)}{(c\tau+d)^{12}\Delta(\tau)} = h_2(\tau)$$

and so

$$u\Big(\frac{a\tau+b}{c\tau+d}\Big) = \frac{h_2(\frac{a\tau+b}{c\tau+d})}{(1+64h_2(\frac{a\tau+b}{c\tau+d}))^2} = \frac{h_2(\tau)}{(1+64h_2(\tau))^2} = u(\tau).$$

By Lemma 4.1, $u(\tau) = u(-\frac{1}{2\tau})$. Therefore, $u(\tau)$ is invariant under $\Gamma_0(2)^+$.

Clearly, $u(\tau) = q + O(q^2)$ and thus has a unique zero at the cusp $[i\infty]$ of order 1, where $q = e^{2\pi i\tau}$. So by the Residue Theorem for compact Riemann surfaces, $u(\tau)$ must be a Hauptmodul for $\Gamma_0(2)^+$. In addition, by the Riemann–Roch theorem, one can show that $2E_2(\tau) - E_2(\tau)$ has a zero of order $\frac{1}{2}$ at the elliptic point $\frac{1+i}{2}$ of $\Gamma_0(2)$, which is of period 2, and so, with respect to $\Gamma_0(2)^+$, it has a zero of order $\frac{1}{4}$ at $\frac{1+i}{2}$, which is an elliptic point of period 4 of $\Gamma_0(2)^+$. The proof is now complete.

Lemma 4.3 *For $a \in \mathbb{Z}$ and $b > 0$ we have $(-1)^a u(\frac{a+\sqrt{-b}}{2}) > 0$.*

Proof. Since $e^{2\pi i\cdot\frac{a+\sqrt{-b}}{2}} = (-1)^a e^{-\sqrt{b}\pi}$, we see that

$$(-1)^a h_2\Big(\frac{a+\sqrt{-b}}{2}\Big) = \frac{e^{-\sqrt{b}\pi}}{\prod_{n=1}^{\infty}(1-(-1)^a e^{-\sqrt{b}\pi(2n-1)})^{24}} > 0$$

and so $(-1)^a u(\frac{a+\sqrt{-b}}{2}) > 0$.

Lemma 4.4 *We have*

$$u\Big(\frac{\sqrt{-2}}{2}\Big) = \frac{1}{256}, \quad u\Big(\frac{1+\sqrt{-3}}{2}\Big) = -\frac{1}{144}, \quad u\Big(\frac{1+i}{4}\Big) = \frac{1}{648},$$
$$u\Big(\frac{1+\sqrt{-7}}{4}\Big) = \frac{1}{81}, \quad u\Big(\frac{1+\sqrt{-7}}{2}\Big) = -\frac{1}{3969}.$$

Proof. Note that $j(\frac{\sqrt{-2}}{2}) = j(\sqrt{-2}) = 20^3$ by [6, (12.20)]. We have $u_1(\frac{\sqrt{-2}}{2})^2 - 207u_1(\frac{\sqrt{-2}}{2}) + 3456 - 20^3 - 20^3 = 0$ by Lemma 4.1. Hence $u_1(\frac{\sqrt{-2}}{2}) \in \{256, -49\}$ and so $u(\frac{\sqrt{-2}}{2}) \in \{\frac{1}{256}, -\frac{1}{49}\}$. Applying Lemma 4.3 gives $u(\frac{\sqrt{-2}}{2}) = \frac{1}{256}$.

By [6, p.261], $j(\frac{1+\sqrt{-3}}{2})=0$ and $j(\sqrt{-3})=54000$. Thus $u_1(\frac{1+\sqrt{-3}}{2})^2-207u_1(\frac{1+\sqrt{-3}}{2})+3456-54000=0$ by Lemma 4.1. Hence $u_1(\frac{1+\sqrt{-3}}{2})\in\{351,-144\}$ and so $u(\frac{1+\sqrt{-3}}{2})\in\{\frac{1}{351},-\frac{1}{144}\}$. In view of Lemma 4.3, $u\big(\frac{1+\sqrt{-3}}{2}\big)=-\frac{1}{144}$.

From (2.3) and [6, (12.20)], $j(\frac{1+i}{2})=j(i-1)=j(i)=12^3$ and $j(\frac{1+i}{4})=j(2i-2)=j(2i)=66^3$. In view of Lemma 4.1, $u_1(\frac{1+i}{4})^2-207u_1(\frac{1+i}{4})+3456-(66^3+12^3)=0$, which yields $u_1(\frac{1+i}{4})\in\{648,-441\}$ and so $u(\frac{1+i}{4})\in\{\frac{1}{648},-\frac{1}{441}\}$. By Lemma 4.1, $u(\frac{1+i}{4})=u(-\frac{1}{(1+i)/2})=u(-1+i)$. Since $e^{2\pi i(-1+i)}=e^{-2\pi}$ we have $h_2(-1+i)=\prod_{n=1}^{\infty}\frac{e^{-2\pi}}{(1-e^{-2\pi(2n-1)})^{24}}>0$ and so $u(\frac{1+i}{4})=u(-1+i)=\frac{h_2(-1+i)}{(1+64h_2(-1+i))^2}>0$. Thus, $u(\frac{1+i}{4})=\frac{1}{648}$.

By [6, (12.20)], $j\big(\frac{3+\sqrt{-7}}{2}\big)=-15^3$. Since $j(\tau+1)=j(\tau)$ and $j(-\frac{1}{\tau})=j(\tau)$, we see that for $\tau_0=\frac{1+\sqrt{-7}}{4}$, $j(\tau_0)=j(2\tau_0-1)=j(2\tau_0)=j(2\tau_0+1)=-15^3$. Applying Lemma 4.1 yields

$$(u_1(\tau_0)-81)(u_1(\tau_0)-126)=u_1(\tau_0)^2-207u_1(\tau_0)+3456+2\cdot 15^3=0.$$

Hence $u_1(\tau_0)\in\{81,126\}$. By (4.4), $4096h_2(\tau_0)$ is a root of $(x+16)^3=-15^3x$. Since $(x+16)^3+15^3x=(x+1)(x^2+47x+4096)$, we see that $4096h_2(\tau_0)\in\{-1,\frac{-47+45\sqrt{-7}}{2},\frac{-47-45\sqrt{-7}}{2}\}$. On the other hand, $4096h_2(\tau_0)+\frac{1}{h_2(\tau_0)}=u_1(\tau_0)-128\in\{-2,-47\}$. Thus, $4096h_2(\tau_0)\neq -1$ and so

$$\begin{aligned}u_1(\tau_0)&=4096h_2(\tau_0)+\frac{1}{h_2(\tau_0)}+128\\&=\frac{-47\pm 45\sqrt{-7}}{2}+\frac{2\cdot 4096}{-47\pm 45\sqrt{-7}}+128=128-47=81,\end{aligned}$$

which yields $u(\tau_0)=\frac{1}{81}$.

Recall that $\tau_0=\frac{1+\sqrt{-7}}{4}$. From [6, (12.20)], $j(4\tau_0)=j(1+\sqrt{-7})=j(\sqrt{-7})=255^3$. By Lemma 4.1, $u_1(2\tau_0)^2-207u_1(2\tau_0)+3456-(-15^3+255^3)=0$, which yields $u_1(2\tau_0)\in\{-3969,4176\}$ and so $u(2\tau_0)\in\{-\frac{1}{3969},\frac{1}{4176}\}$. Applying Lemma 4.3, $u(2\tau_0)<0$. Thus $u(2\tau_0)=-\frac{1}{3969}$. The proof is now complete.

Remark 4.1 Lemma 4.4 was stated by Beukers in [2, p.30] without proof.

By Lemma 4.2, both $u(\tau)$ and $\sum_{n=0}^{\infty}\binom{2n}{n}^2\binom{4n}{2n}u(\tau)^n$ satisfy the assumptions in Beukers' theorem.

Proof of (1.7). Since $p=x^2+7y^2=(x+y\sqrt{-7})(x-y\sqrt{-7})$ for $x,y\in\mathbb{Z}$, one may choose the sign of y so that $x+y\sqrt{-7}\notin p\mathbb{Z}_p$. Set $a_n=\binom{2n}{n}^2\binom{4n}{2n}$, $c=$

$2y,\ d=x-y,\ \alpha=\frac{1+\sqrt{-7}}{2}$ and $N=2$. Then clearly (2.4) holds. By Lemma 4.4, $u(\frac{1+\sqrt{-7}}{2})=-\frac{1}{3969}$. Hence, applying Beukers' theorem(i) and Lemma 2.1 we obtain

$$\begin{aligned}\sum_{n=0}^{p-1}\frac{\binom{2n}{n}^2\binom{4n}{2n}}{(-3969)^n}&=\sum_{n=0}^{p-1}\binom{2n}{n}^2\binom{4n}{2n}u\Big(\frac{1+\sqrt{-7}}{2}\Big)^n\equiv\Big(2y\cdot\frac{1+\sqrt{-7}}{2}+x-y\Big)^2\\&=(x+y\sqrt{-7})^2\equiv 4x^2-2p-\frac{p^2}{4x^2}\pmod{p^3}.\end{aligned}$$

On the other hand, setting $a_n=\binom{2n}{n}^2\binom{4n}{2n},\ c=4y,\ d=x-y,\ \alpha=\frac{1+\sqrt{-7}}{4}$ and $N=2$ one finds that (2.4) holds. By Lemma 4.4, $u(\frac{1+\sqrt{-7}}{4})=\frac{1}{81}$. Hence, applying Beukers' theorem(i) and Lemma 2.1 we obtain

$$\begin{aligned}\sum_{n=0}^{p-1}\frac{\binom{2n}{n}^2\binom{4n}{2n}}{81^n}&=\sum_{n=0}^{p-1}\binom{2n}{n}^2\binom{4n}{2n}u\Big(\frac{1+\sqrt{-7}}{4}\Big)^n\equiv\Big(4y\cdot\frac{1+\sqrt{-7}}{4}+x-y\Big)^2\\&=(x+y\sqrt{-7})^2\equiv 4x^2-2p-\frac{p^2}{4x^2}\pmod{p^3}.\end{aligned}$$

Proof of (1.9). Since $p=x^2+3y^2=(x+y\sqrt{-3})(x-y\sqrt{-3})$ for $x,y\in\mathbb{Z}$, one may choose the sign of y so that $x+y\sqrt{-3}\notin p\mathbb{Z}_p$. Set $a_n=\binom{2n}{n}^2\binom{4n}{2n},\ c=2y,\ d=x-y,\ \alpha=\frac{1+\sqrt{-3}}{2}$ and $N=2$. Then clearly (2.4) holds. By Lemma 4.4, $u(\frac{1+\sqrt{-3}}{2})=-\frac{1}{144}$. Hence, applying Beukers' theorem(i) and Lemma 2.1 we obtain

$$\begin{aligned}\sum_{n=0}^{p-1}\frac{\binom{2n}{n}^2\binom{4n}{2n}}{(-144)^n}&=\sum_{n=0}^{p-1}\binom{2n}{n}^2\binom{4n}{2n}u\Big(\frac{1+\sqrt{-3}}{2}\Big)^n\equiv\Big(2y\cdot\frac{1+\sqrt{-3}}{2}+x-y\Big)^2\\&=(x+y\sqrt{-3})^2\equiv 4x^2-2p-\frac{p^2}{4x^2}\pmod{p^3}.\end{aligned}$$

Proof of (1.11). Since $p=x^2+4y^2=(x+2yi)(x-2yi)$ for $x,y\in\mathbb{Z}$, one may choose the sign of y so that $x+2yi\notin p\mathbb{Z}_p$. Set $a_n=\binom{2n}{n}^2\binom{4n}{2n},\ c=8y,\ d=x-2y,\ \alpha=\frac{1+i}{4}$ and $N=2$. One can check that (2.4) holds. By Lemma 4.4, $u(\frac{1+i}{4})=\frac{1}{648}$. Hence, applying Beukers' theorem(i) and Lemma 2.1 we obtain

$$\begin{aligned}\sum_{n=0}^{p-1}\frac{\binom{2n}{n}^2\binom{4n}{2n}}{648^n}&=\sum_{n=0}^{p-1}\binom{2n}{n}^2\binom{4n}{2n}u\Big(\frac{1+i}{4}\Big)^n\equiv\Big(8y\cdot\frac{1+i}{4}+x-2y\Big)^2\\&=(x+y\sqrt{-4})^2\equiv 4x^2-2p-\frac{p^2}{4x^2}\pmod{p^3}.\end{aligned}$$

Proof of (1.13). Since $p=x^2+2y^2=(x+y\sqrt{-2})(x-y\sqrt{-2})$ for $x,y\in\mathbb{Z}$, one may choose the sign of y so that $x+y\sqrt{-2}\notin p\mathbb{Z}_p$. Set $a_n=\binom{2n}{n}^2\binom{4n}{2n},\ c=2y,\ d=x,\ \alpha=\frac{\sqrt{-2}}{2}$ and $N=2$. Then clearly (2.4) holds. By Lemma 4.4, $u(\frac{\sqrt{-2}}{2})=\frac{1}{256}$. Hence, applying Beukers' theorem(i) and Lemma 2.1 we obtain

$$\begin{aligned}\sum_{n=0}^{p-1}\frac{\binom{2n}{n}^2\binom{4n}{2n}}{256^n}&=\sum_{n=0}^{p-1}\binom{2n}{n}^2\binom{4n}{2n}u\Big(\frac{\sqrt{-2}}{2}\Big)^n\equiv\Big(2y\cdot\frac{\sqrt{-2}}{2}+x\Big)^2\\&\equiv 4x^2-2p-\frac{p^2}{4x^2}\pmod{p^3}.\end{aligned}$$

Lemma 4.5 *We have*

$$\begin{aligned}&u\Big(\frac{\sqrt{-6}}{2}\Big)=\frac{1}{48^2},\ u\Big(\frac{\sqrt{-10}}{2}\Big)=\frac{1}{12^4},\ u\Big(\frac{3\sqrt{-2}}{2}\Big)=\frac{1}{28^4},\\&u\Big(\frac{\sqrt{-22}}{2}\Big)=\frac{1}{1584^2},\ \ u\Big(\frac{\sqrt{-58}}{2}\Big)=\frac{1}{396^4},\\&u\Big(\frac{1+3i}{2}\Big)=-\frac{1}{12288},\ u\Big(\frac{1+5i}{2}\Big)=-\frac{1}{6635520},\ u\Big(\frac{1+\sqrt{-5}}{2}\Big)=-\frac{1}{1024},\\&u\Big(\frac{1+\sqrt{-13}}{2}\Big)=-\frac{1}{82944},\ u\Big(\frac{1+\sqrt{-37}}{2}\Big)=-\frac{1}{14112^2}.\end{aligned}$$

Proof. From [1, pp.200-201], for $n=6,10,18,22,58$ we have

$$h_2\Big(\frac{\sqrt{-n}}{2}\Big)=\frac{\eta(\sqrt{-n})^{24}}{\eta(\frac{\sqrt{-n}}{2})^{24}}=\begin{cases}\frac{1}{64(1+\sqrt{2})^4}=\frac{1}{64(17+12\sqrt{2})} & \text{if } n=6,\\ \frac{1}{64(\frac{1+\sqrt{5}}{2})^{12}}=\frac{1}{64(161+72\sqrt{5})} & \text{if } n=10,\\ \frac{1}{64(\sqrt{2}+\sqrt{3})^8}=\frac{1}{64(4801+1960\sqrt{6})} & \text{if } n=18,\\ \frac{1}{64(1+\sqrt{2})^{12}}=\frac{1}{64(19601+13860\sqrt{2})} & \text{if } n=22,\\ \frac{1}{64(\frac{5+\sqrt{29}}{2})^{12}}=\frac{1}{64(192119201+35675640\sqrt{29})} & \text{if } n=58.\end{cases}\qquad(4.5)$$

Substituting these values into the formula $u\big(\frac{\sqrt{-n}}{2}\big)=\frac{h_2(\frac{\sqrt{-n}}{2})}{(1+64h_2(\frac{\sqrt{-n}}{2}))^2}$ yields the results for $u\big(\frac{\sqrt{-n}}{2}\big)$ in the cases $n=6,10,18,22,58$.

From [1, pp.189-191], for $n=5,9,13,25,37$ we have

$$h_2\Big(\frac{1+\sqrt{-n}}{2}\Big)=\frac{\eta(1+\sqrt{-n})^{24}}{\eta(\frac{1+\sqrt{-n}}{2})^{24}}=\frac{\eta(\sqrt{-n})^{24}}{\eta(\frac{1+\sqrt{-n}}{2})^{24}}=\begin{cases}-\frac{1}{64(\frac{1+\sqrt5}{2})^6} & \text{if } n=5,\\ -\frac{1}{64(\frac{1+\sqrt3}{\sqrt2})^8} & \text{if } n=9,\\ -\frac{1}{64(\frac{3+\sqrt{13}}{2})^6} & \text{if } n=13,\\ -\frac{1}{64(\frac{1+\sqrt5}{2})^{24}} & \text{if } n=25,\\ -\frac{1}{64(6+\sqrt{37})^6} & \text{if } n=37.\end{cases} \tag{4.6}$$

Substituting these values into the formula $u\big(\frac{1+\sqrt{-n}}{2}\big)=\frac{h_2(\frac{1+\sqrt{-n}}{2})}{(1+64h_2(\frac{1+\sqrt{-n}}{2}))^2}$ yields the results for $u\big(\frac{1+\sqrt{-n}}{2}\big)$ in the cases $n=5,9,13,25,37$. The proof is now complete.

Remark 4.2 Lemma 4.5 was stated by Beukers in [2, p.30] without proof. By (4.4),

$$j(\tau)=\frac{(1+256h_2(\tau))^3}{h_2(\tau)}\quad\text{and}\quad j(2\tau)=h_2(\tau)\Big(16+\frac{1}{h_2(\tau)}\Big)^3.$$

This together with (4.5) and (4.6) yields the values for $j(\frac{\sqrt{-n}}{2})$ and $j(\sqrt{-n})$ in the cases $n=6,10,18,22,58$, and $j(\frac{1+\sqrt{-n}}{2})$ and $j(\sqrt{-n})$ in the cases $n=5,9,13,25,37$.

Let $u(\tau)$ be given in (4.1). By Lemma 4.2, both $u(\tau)$ and $\sum_{n=0}^\infty\binom{2n}{n}^2\binom{4n}{2n}u(\tau)^n$ satisfy the assumptions in Beukers' theorem.

Proof of (1.14). Since $p=x^2+2y^2=(x+y\sqrt{-2})(x-y\sqrt{-2})$ for $x,y\in\mathbb{Z}$, one may choose the sign of y so that $x+y\sqrt{-2}\notin p\mathbb{Z}_p$. For $p\equiv1\pmod 3$ we have $3\mid y$. Set $a_n=\binom{2n}{n}^2\binom{4n}{2n},\ c=\frac{2y}{3},\ d=x,\ \alpha=\frac{3\sqrt{-2}}{2}$ and $N=2$. One can find that (2.4) holds. By Lemma 4.5, $u(\frac{3\sqrt{-2}}{2})=\frac{1}{28^4}$. Hence, applying Beukers' theorem(i) and Lemma 2.1 we obtain

$$\begin{aligned}\sum_{n=0}^{p-1}\frac{\binom{2n}{n}^2\binom{4n}{2n}}{28^{4n}}&=\sum_{n=0}^{p-1}\binom{2n}{n}^2\binom{4n}{2n}u\Big(\frac{3\sqrt{-2}}{2}\Big)^n\equiv\Big(\frac{2y}{3}\cdot\frac{3\sqrt{-2}}{2}+x\Big)^2\\&\equiv 4x^2-2p-\frac{p^2}{4x^2}\pmod{p^3}.\end{aligned}$$

Now assume $p\equiv2\pmod 3$. Then $3\mid x$. Setting $a_n=\binom{2n}{n}^2\binom{4n}{2n},\ c=\frac{x}{3},\ d=-y,\ \alpha=\frac{3\sqrt{-2}}{2}$ and $N=2$, we find that (2.5) holds. Since $u\big(\frac{3\sqrt{-2}}{2}\big)=\frac{1}{28^4}$ by Lemma

4.5, using Beukers' theorem(ii) and Lemma 2.1 we deduce that

$$\sum_{n=0}^{p-1}\frac{\binom{2n}{n}^2\binom{4n}{2n}}{28^{4n}}=\sum_{n=0}^{p-1}\binom{2n}{n}^2\binom{4n}{2n}u\Big(\frac{3\sqrt{-2}}{2}\Big)^n\equiv -2\Big(\frac{x}{3}\cdot\frac{3\sqrt{-2}}{2}-y\Big)^2$$
$$=(x+\sqrt{-2}y)^2\equiv 4x^2-2p-\frac{p^2}{4x^2}\pmod{p^3}.$$

Proof of Theorems 1.5-1.6. Let $m\in\{5,9,13,25,37\}$ and $D(m)=-1024$, $-12288,-82944,-6635520,-14112^2$ according as $m=5,9,13,25,37$. For $p=x^2+my^2$ we have $p=(x+y\sqrt{-m})(x-y\sqrt{-m})$ for $x,y\in\mathbb{Z}$, one may choose the sign of y so that $x+y\sqrt{-m}\notin p\mathbb{Z}_p$. Set $a_n=\binom{2n}{n}^2\binom{4n}{2n},\ c=2y,\ d=x-y,\ \alpha=\frac{1+\sqrt{-m}}{2}$ and $N=2$. Then clearly (2.4) holds. By Lemma 4.5, $u(\frac{1+\sqrt{-m}}{2})=\frac{1}{D(m)}$. Hence, applying Beukers' theorem(i) and Lemma 2.1 we obtain

$$\sum_{n=0}^{p-1}\frac{\binom{2n}{n}^2\binom{4n}{2n}}{D(m)^n}=\sum_{n=0}^{p-1}\binom{2n}{n}^2\binom{4n}{2n}u\Big(\frac{1+\sqrt{-m}}{2}\Big)^n\equiv\Big(2y\cdot\frac{1+\sqrt{-m}}{2}+x-y\Big)^2$$
$$=(x+\sqrt{-m}y)^2\equiv 4x^2-2p-\frac{p^2}{4x^2}\pmod{p^3}.$$

For $2p=x^2+my^2$ we have $2\nmid xy$. One may choose the sign of y so that $x+y\sqrt{-m}\notin p\mathbb{Z}_p$. Setting $a_n=\binom{2n}{n}^2\binom{4n}{2n},\ c=y,\ d=\frac{x-y}{2},\ \alpha=\frac{1+\sqrt{-m}}{2}$ and $N=2$, one finds that (2.5) holds. Since $u\big(\frac{1+\sqrt{-m}}{2}\big)=\frac{1}{D(m)}$ by Lemma 4.5, using Beukers' theorem(ii) and Lemma 2.1 we deduce that

$$\sum_{n=0}^{p-1}\frac{\binom{2n}{n}^2\binom{4n}{2n}}{D(m)^n}=\sum_{n=0}^{p-1}\binom{2n}{n}^2\binom{4n}{2n}u\Big(\frac{1+\sqrt{-m}}{2}\Big)^n\equiv -2\Big(y\cdot\frac{1+\sqrt{-m}}{2}+\frac{x-y}{2}\Big)^2$$
$$=-\frac{1}{2}(x+\sqrt{-m}y)^2\equiv-\frac{1}{2}\Big(4x^2-4p-\frac{p^2}{x^2}\Big)$$
$$=-2x^2+2p+\frac{p^2}{2x^2}\pmod{p^3}.$$

Proof of Theorem 1.7. For $p=x^2+2my^2$ we have $p=(x+y\sqrt{-2m})(x-y\sqrt{-2m})$ for $x,y\in\mathbb{Z}$, one may choose the sign of y so that $x+y\sqrt{-2m}\notin p\mathbb{Z}_p$. Set $a_n=\binom{2n}{n}^2\binom{4n}{2n},\ c=2y,\ d=x,\ \alpha=\frac{\sqrt{-2m}}{2}$ and $N=2$. Then clearly (2.4) holds. By Lemma 4.5, $u(\frac{\sqrt{-2m}}{2})=\frac{1}{F(m)}$. Hence, applying Beukers' theorem(i) and Lemma 2.1

we obtain

$$\sum_{n=0}^{p-1}\frac{\binom{2n}{n}^2\binom{4n}{2n}}{F(m)^n}=\sum_{n=0}^{p-1}\binom{2n}{n}^2\binom{4n}{2n}u\Big(\frac{\sqrt{-2m}}{2}\Big)^n\equiv\Big(2y\cdot\frac{\sqrt{-2m}}{2}+x\Big)^2$$
$$=(x+\sqrt{-2m}y)^2\equiv 4x^2-2p-\frac{p^2}{4x^2}\pmod{p^3}.$$

For $p=2x^2+my^2$ we have $2\nmid y$ and $2p=(2x)^2+2my^2=(2x+y\sqrt{-2m})(2x-y\sqrt{-2m})$. One may choose the sign of y so that $2x+y\sqrt{-2m}\notin p\mathbb{Z}_p$. Setting $a_n=\binom{2n}{n}^2\binom{4n}{2n}$, $c=y$, $d=x$, $\alpha=\frac{\sqrt{-2m}}{2}$ and $N=2$, we find that (2.5) holds. Since $u\big(\frac{\sqrt{-2m}}{2}\big)=\frac{1}{F(m)}$ by Lemma 4.5, using Beukers' theorem(ii) and Lemma 2.1 we deduce that

$$\sum_{n=0}^{p-1}\frac{\binom{2n}{n}^2\binom{4n}{2n}}{F(m)^n}=\sum_{n=0}^{p-1}\binom{2n}{n}^2\binom{4n}{2n}u\Big(\frac{\sqrt{-2m}}{2}\Big)^n\equiv-2\Big(y\cdot\frac{\sqrt{-2m}}{2}+x\Big)^2$$
$$=-\frac{1}{2}(2x+\sqrt{-2m}y)^2\equiv-\frac{1}{2}\Big(4(2x)^2-4p-\frac{4p^2}{4(2x)^2}\Big)$$
$$=-8x^2+2p+\frac{p^2}{8x^2}\pmod{p^3}.$$

5. Proofs of congruences involving $\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}$

For given positive integer k let $(a)_k=a(a+1)\cdots(a+k-1)$. Then $(a)_k=(-1)^k\binom{-a}{k}k!$. From [18] we know that

$$\frac{\big(\frac12\big)_k\big(\frac16\big)_k\big(\frac56\big)_k}{(1)_k^3}=(-1)^k\binom{-\frac12}{k}\binom{-\frac16}{k}\binom{-\frac56}{k}=\frac{\binom{2k}{k}}{4^k}\cdot\frac{\binom{3k}{k}\binom{6k}{3k}}{432^k}=\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}}.$$

Proof of Theorem 1.8. Suppose that $p=4n+1=x^2+4y^2$ so that $x+y\sqrt{-4}\notin p\mathbb{Z}_p$. Taking $D=4$ in [2, Theorem 1.27] and $d_K=-4$ in [6, (12.20)] gives $\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}12^{-3k}\equiv\pm(x+y\sqrt{-4})^2\pmod{p^3}$. Applying Lemma 2.1 we get

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}}\equiv\pm\Big(4x^2-2p-\frac{p^2}{4x^2}\Big)\pmod{p^3}.$$

Since Mortenson[12] proved that $\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}12^{-3k} \equiv \left(\frac{-3}{p}\right)4x^2 \pmod{p^2}$, we must have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}} \equiv \left(\frac{-3}{p}\right)\Big(4x^2-2p-\frac{p^2}{4x^2}\Big) \pmod{p^3}.$$

On the other hand, taking $D=16$ in [2, Theorem 1.27] and $d_K=-16$ in [6, (12.20)] gives $\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}66^{-3k} \equiv \pm(x+y\sqrt{-4})^2 \pmod{p^3}$. Applying Lemma 2.1 we get

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{66^{3k}} \equiv \pm\Big(4x^2-2p-\frac{p^2}{4x^2}\Big) \pmod{p^3}.$$

By [18, Theorem 4.3], $\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}66^{-3k} \equiv \left(\frac{33}{p}\right)4x^2 \pmod{p}$. Hence,

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{66^{3k}} \equiv \left(\frac{33}{p}\right)\Big(4x^2-2p-\frac{p^2}{4x^2}\Big) \pmod{p^3}.$$

Proof of Theorem 1.9. Suppose that $p=3n+1=x^2+3y^2 (x,y\in\mathbb{Z})$ and we choose the sign of y so that $x+y\sqrt{-3}\notin p\mathbb{Z}_p$. Taking $D=12$ in [2, Theorem 1.27] and $d_K=-12$ in [6, (12.20)] gives $\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}54000^{-k} \equiv \pm(x+y\sqrt{-3})^2 \pmod{p^3}$. Applying Lemma 2.1 we get

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{54000^{k}} \equiv \pm\Big(4x^2-2p-\frac{p^2}{4x^2}\Big) \pmod{p^3}.$$

By [18, Theorem 4.5], $\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}54000^{-k} \equiv \left(\frac{5}{p}\right)4x^2 \pmod{p}$. Hence,

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{54000^{k}} \equiv \left(\frac{5}{p}\right)\Big(4x^2-2p-\frac{p^2}{4x^2}\Big) \pmod{p^3}.$$

Proof of Theorem 1.13 in the case $m=11$. Suppose that $\left(\frac{p}{11}\right)=1$ and so $4p=x^2+11y^2 (x,y\in\mathbb{Z})$. We choose the sign of y so that $x+y\sqrt{-11}\notin p\mathbb{Z}_p$. Taking $D=11$ in [2, Theorem 1.27] and $d_K=-11$ in [6, (12.20)] gives $\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}(-32)^{-3k} \equiv \pm\left(\frac{x+y\sqrt{-11}}{2}\right)^2 \pmod{p^3}$. Applying Lemma 2.1 (with $c=4$) we get

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-32)^{3k}} \equiv \pm\frac{1}{4}\Big(4x^2-2\cdot 4p-\frac{4^2p^2}{4x^2}\Big)=\pm\Big(x^2-2p-\frac{p^2}{x^2}\Big) \pmod{p^3}.$$

By [18, Theorem 4.8], $\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}(-32)^{-3k} \equiv \left(\frac{-2}{p}\right)x^2 \pmod p$. Hence,

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-32)^{3k}} \equiv \left(\frac{-2}{p}\right)\left(x^2-2p-\frac{p^2}{x^2}\right) \pmod{p^3}.$$

In a similar way, using [2, Theorem 1.27], [6, (12.20)] and [18, Theorems 4.4, 4.6, 4.7 and 4.9] one can prove Theorems 1.10-1.12 and Theorem 1.13 in the cases $m = 19, 43, 67, 163$.

Declaration of competing interest.
The authors declare no conflicts of interest regarding the publication of this paper.

Data availability.
No data was used for the research described in the article

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