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Congruences for Bernoulli numbers and Bernoulli polynomials

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Abstract

Let $\{B_n(x)\}$ be the well-known Bernoulli polynomials. It is the purpose of this paper to determine $pB_{k(p-1)+b}(x) \bmod p^n$, where p is a prime, k, b nonnegative integers and x a rational p-integer.

1. Introduction

The Bernoulli numbers $\{B_n\}$ and Bernoulli polynomials $\{B_n(x)\}$ are given by

$$B_0 = 1, \quad \sum_{k=0}^{n-1} \binom{n}{k} B_k = 0 \quad (n = 2, 3, 4, \dots)$$

and

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} x^k.$$

It is interesting to investigate arithmetic properties of $\{B_n\}$ and $\{B_n(x)\}$. For the work on this line one may consult [1–3, 5–9, 11, 12]. Here we give two classical results (cf. [8]):

Kummer's congruences: Let p be an odd prime, and b an even number with $p-1 \nmid b$. For $k = 0, 1, 2, \dots$ we have

$$\frac{B_{k(p-1)+b}}{k(p-1)+b} \equiv \frac{B_b}{b} \pmod p.$$

Von Staudt–Clausen Theorem: Suppose that p is a prime and $k \in \mathbb{Z}^+$. Then

$$pB_{k(p-1)} \equiv \begin{cases} 0 \pmod p & \text{if } p = 2 \text{ and } k > 1 \text{ is odd,} \\ -1 \pmod p & \text{otherwise.} \end{cases}$$

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In [10] the author proved that

$$pB_{k(p-1)} \equiv kpB_{p-1} - (k-1)(p-1) \pmod{p^2},$$

$$pB_{k(p-1)} \equiv \binom{k}{2} pB_{2p-2} - k(k-2)pB_{p-1} + \binom{k-1}{2}(p-1) \pmod{p^3},$$

where $p > 3$ is a prime and $k \in \mathbb{Z}^+$.

Let p be a prime, and $\mathbb{Z}_p$ the set of rational p-integers. Suppose that $b \in \mathbb{Z}^+ \cup \{0\}$ and $x \in \mathbb{Z}_p$. In this paper we determine $pB_{k(p-1)+b}(x) \bmod p^n$ by proving that

$$pB_{k(p-1)+b}(x) - p^{k(p-1)+b} B_{k(p-1)+b}\left(\frac{x + \langle -x \rangle_p}{p}\right)$$

$$\equiv \sum_{r=0}^{n-1} (-1)^{n-1-r} \binom{k-1-r}{n-1-r} \binom{k}{r} \Bigg(pB_{r(p-1)+b}(x) - p^{r(p-1)+b}$$

$$\times B_{r(p-1)+b}\left(\frac{x + \langle -x \rangle_p}{p}\right)\Bigg) + (-1)^n \delta(n, b, p) \binom{k}{n} p^{n-1}$$

$$\equiv a_n k^n + a_{n-1} k^{n-1} + \cdots + a_1 k + a_0 \pmod{p^n} \quad (k = 0, 1, 2, \ldots),$$

where $a_0, a_1, \ldots, a_n$ are all integers, $\langle -x \rangle_p$ is the least nonnegative residue of $-x \bmod p$, and

$$\delta(n, b, p) = \begin{cases} 1 & \text{if } B_n \notin \mathbb{Z}_p \text{ and } p - 1 \mid b, \\ 0 & \text{if } B_n \in \mathbb{Z}_p \text{ or } p - 1 \nmid b. \end{cases} \tag{1.1}$$

Clearly, our results provide a wide generalization of von Staudt–Clausen theorem.

2. Basic lemmas

In this section we give several lemmas which will be used later.

Lemma 2.1. *Let $n \geqslant 1$ and $k \geqslant 0$ be integers. For any function f we have*

$$f(k) = \sum_{r=0}^{n-1} (-1)^{n-1-r} \binom{k-1-r}{n-1-r} \binom{k}{r} f(r) + \sum_{r=n}^{k} \binom{k}{r} (-1)^r \sum_{s=0}^{r} \binom{r}{s} (-1)^s f(s),$$

where the second sum is zero when $k < n$.

Proof. Note that

$$\sum_{j=0}^{m}(-1)^j\binom{x}{j}=\sum_{j=0}^{m}(-1)^j\binom{x-1}{j}+\sum_{j=1}^{m}(-1)^j\binom{x-1}{j-1}$$
$$=\sum_{r=0}^{m}(-1)^r\binom{x-1}{r}-\sum_{r=0}^{m-1}(-1)^r\binom{x-1}{r}$$
$$=(-1)^m\binom{x-1}{m},$$

we find

$$\sum_{r=0}^{n-1}\binom{k}{r}(-1)^r\sum_{s=0}^{r}\binom{r}{s}(-1)^s f(s)=\sum_{s=0}^{n-1}\sum_{r=s}^{n-1}\binom{k}{r}\binom{r}{s}(-1)^{r-s}f(s)$$
$$=\sum_{s=0}^{n-1}\binom{k}{s}\sum_{r=s}^{n-1}\binom{k-s}{r-s}(-1)^{r-s}f(s)=\sum_{s=0}^{n-1}\binom{k}{s}\sum_{j=0}^{n-1-s}\binom{k-s}{j}(-1)^j f(s)$$
$$=\sum_{s=0}^{n-1}(-1)^{n-1-s}\binom{k-1-s}{n-1-s}\binom{k}{s}f(s).$$

Now applying binomial inversion formula yields the result. □

Lemma 2.2 (Raabe's theorem (cf. Ireland and Rosen [8])). *Let $m,n\in\mathbb{Z}^+$ and $x\in\mathbb{R}$. Then*

$$m^{n-1}\sum_{r=0}^{m-1}B_n\left(x+\frac{r}{m}\right)=B_n(mx).$$

Lemma 2.3. *Suppose that $p>1$ and $k\geqslant 1$ are integers. If $x, x_0\in\mathbb{Z}_p$ and $x\equiv x_0 \pmod p$ then*

$$pB_k(x),\ \frac{B_k(x)-B_k(x_0)}{k},\ \frac{B_k(x)-B_k}{k}\in\mathbb{Z}_p.$$

Proof. For $n\geqslant 1$ we have pB_n, $p^{n-1}/n\in\mathbb{Z}_p$. Thus,

$$pB_k(x)=\sum_{r=0}^{k}\binom{k}{r}pB_{k-r}x^r\in\mathbb{Z}_p.$$

It is well known that

$$B_k(x_0+t)=\sum_{r=0}^{k}\binom{k}{r}B_{k-r}(x_0)t^r.$$

So

$$\frac{B_k(x)-B_k(x_0)}{k}=\sum_{r=1}^{k}\binom{k-1}{r-1}B_{k-r}(x_0)\frac{(x-x_0)^r}{r}$$
$$=\sum_{r=1}^{k}\binom{k-1}{r-1}pB_{k-r}(x_0)\left(\frac{x-x_0}{p}\right)^r\cdot\frac{p^{r-1}}{r}\in\mathbb{Z}_p.$$

Assume $x\equiv n\pmod p$ for some $n\in\mathbb{Z}^+$. By the above and the fact $\sum_{r=0}^{n-1}r^{k-1}=(B_k(n)-B_k)/k$ we obtain

$$\frac{B_k(x)-B_k}{k}=\frac{B_k(x)-B_k(n)}{k}+\frac{B_k(n)-B_k}{k}\in\mathbb{Z}_p,$$

which completes the proof. □

3. Congruences of the Kummer type

In this section we prove the following.

Theorem 3.1. *Let p be a prime, $n\in\mathbb{Z}^+$, $b\in\mathbb{Z}^+\cup\{0\}$ and $x\in\mathbb{Z}_p$. Then*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\left(pB_{k(p-1)+b}(x)-p^{k(p-1)+b}B_{k(p-1)+b}\left(\frac{x+\langle -x\rangle_p}{p}\right)\right)$$
$$\equiv\begin{cases}0\pmod{p^n} & \text{if } n\in S_p \text{ or } p-1\nmid b,\\ p^{n-1}\pmod{p^n} & \text{if } n\notin S_p \text{ and } p-1\mid b,\end{cases}$$

where

$$S_p=\{m\mid B_m\in\mathbb{Z}_p,\ m\in\mathbb{Z}^+\}$$
$$=\begin{cases}\{3,5,7,\ldots\} & \text{if } p=2,\\ \{m\mid p-1\nmid m,\ m\in\mathbb{Z}^+\} & \text{if } p>2\end{cases}$$

and $\langle -x\rangle_p$ denotes the least nonnegative residue of $-x$ modulo p.

Proof. Since

$$B_n(x)=\sum_{r=0}^{n}\binom{n}{r}B_rx^{n-r}$$

one has

$$B_{k(p-1)+b}\left(\frac{x}{p}+\frac{j}{p}\right)=\sum_{r=0}^{k(p-1)+b}\binom{k(p-1)+b}{r}\left(\frac{x+j}{p}\right)^{k(p-1)+b-r}B_r.$$

Using Raabe's theorem one sees that

$$p^{-k(p-1)-b+1}B_{k(p-1)+b}(x) - B_{k(p-1)+b}\left(\frac{x+\langle -x\rangle_p}{p}\right)$$
$$= \sum_{j=0}^{p-1} B_{k(p-1)+b}\left(\frac{x}{p}+\frac{j}{p}\right) - B_{k(p-1)+b}\left(\frac{x+\langle -x\rangle_p}{p}\right)$$
$$= \sum_{r=0}^{k(p-1)+b} \binom{k(p-1)+b}{r} \sum_{\substack{j=0\\ j\neq \langle -x\rangle_p}}^{p-1} \left(\frac{x+j}{p}\right)^{k(p-1)+b-r} B_r,$$

which gives

$$pB_{k(p-1)+b}(x) - p^{k(p-1)+b}B_{k(p-1)+b}\left(\frac{x+\langle -x\rangle_p}{p}\right)$$
$$= \sum_{r=0}^{k(p-1)+b} \binom{k(p-1)+b}{r} \sum_{\substack{j=0\\ j\neq \langle -x\rangle_p}}^{p-1} (x+j)^{k(p-1)+b-r}p^rB_r.$$

It then follows that

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\left(pB_{k(p-1)+b}(x) - p^{k(p-1)+b}B_{k(p-1)+b}\left(\frac{x+\langle -x\rangle_p}{p}\right)\right)$$
$$= \sum_{k=0}^{n}\sum_{r=0}^{k(p-1)+b}\binom{n}{k}(-1)^k\binom{k(p-1)+b}{r}\sum_{\substack{j=0\\ j\neq \langle -x\rangle_p}}^{p-1} (x+j)^{k(p-1)+b-r}p^rB_r$$
$$= \sum_{r=0}^{n(p-1)+b} p^rB_r \sum_{\substack{j=0\\ j\neq \langle -x\rangle_p}}^{p-1} \sum_{k=0}^{n}\binom{n}{k}(-1)^k\binom{k(p-1)+b}{r}(x+j)^{k(p-1)+b-r}.$$

Let $f(k)$ be a polynomial of degree r with the property $f(k)\in\mathbb{Z}$ for $k\in\mathbb{Z}$. It is well known that $f(k)=\sum_{s=0}^{r} a_s\binom{k}{s}$ for some $a_0, \ldots, a_r\in\mathbb{Z}$.

Now, letting $f(k)=\binom{k(p-1)+b}{r}$ we get

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\binom{k(p-1)+b}{r}(x+j)^{k(p-1)+b-r}$$
$$= \sum_{k=0}^{n}\sum_{s=0}^{r}\binom{n}{k}(-1)^k a_s\binom{k}{s}(x+j)^{k(p-1)+b-r}$$
$$= \sum_{s=0}^{r}(-1)^s a_s\binom{n}{s}\sum_{k=s}^{n}(-1)^{k-s}\binom{n-s}{k-s}(x+j)^{k(p-1)+b-r}$$
$$\left(\text{observing that } \binom{n}{k}\binom{k}{s}=\binom{n}{s}\binom{n-s}{k-s}\right)$$
$$= \sum_{s=0}^{r}(-1)^s a_s\binom{n}{s}(x+j)^{s(p-1)+b-r}(1-(x+j)^{p-1})^{n-s}.$$

So

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\left(pB_{k(p-1)+b}(x)-p^{k(p-1)+b}B_{k(p-1)+b}\left(\frac{x+\langle -x\rangle_p}{p}\right)\right)$$

$$=\sum_{r=0}^{n(p-1)+b}p^rB_r\sum_{\substack{j=0\\ j\neq\langle -x\rangle_p}}^{p-1}\sum_{s=0}^{r}(-1)^s a_s\binom{n}{s}(x+j)^{s(p-1)+b-r}(1-(x+j)^{p-1})^{n-s}.$$

Since

$$p^rB_r\cdot(1-(x+j)^{p-1})^{n-s}=p^{n-s+r-1}\cdot pB_r\cdot\left(\frac{1-(x+j)^{p-1}}{p}\right)^{n-s}\equiv 0(\operatorname{mod} p^n)$$

for $j\neq\langle -x\rangle_p$ and $s\in\{0,1,\dots,r-1\}$ we see that

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\left(pB_{k(p-1)+b}(x)-p^{k(p-1)+b}B_{k(p-1)+b}\left(\frac{x+\langle -x\rangle_p}{p}\right)\right)$$

$$\equiv\sum_{r=0}^{n(p-1)+b}p^nB_r\sum_{\substack{j=0\\ j\neq\langle -x\rangle_p}}^{p-1}(-1)^r a_r\binom{n}{r}(x+j)^{r(p-1)+b-r}\cdot\left(\frac{1-(x+j)^{p-1}}{p}\right)^{n-r}$$

$$\equiv\sum_{r=0}^{n}p^nB_r\sum_{\substack{j=0\\ j\neq\langle -x\rangle_p}}^{p-1}\binom{n}{r}(x+j)^{b-r}\cdot\left(\frac{1-(x+j)^{p-1}}{p}\right)^{n-r}$$

(using the fact $a_r=(p-1)^r\equiv(-1)^r(\operatorname{mod} p)$ and Fermat's little theorem)

$$=\sum_{\substack{j=0\\ j\neq\langle -x\rangle_p}}^{p-1}(x+j)^{b-n}p^n\sum_{r=0}^{n}\binom{n}{r}B_r\left((x+j)\cdot\frac{1-(x+j)^{p-1}}{p}\right)^{n-r}$$

$$=\sum_{\substack{j=0\\ j\neq\langle -x\rangle_p}}^{p-1}(x+j)^{b-n}p^nB_n\left(\frac{(x+j)-(x+j)^p}{p}\right)$$

$$=\sum_{\substack{j=0\\ j\neq\langle -x\rangle_p}}^{p-1}(x+j)^{b-n}\left(p^nB_n\left(\frac{(x+j)-(x+j)^p}{p}\right)-p^nB_n+p^nB_n\right)$$

$$\equiv p^nB_n\sum_{\substack{j=0\\ j\neq\langle -x\rangle_p}}^{p-1}(x+j)^{b-n}$$

$$\left(\text{since } B_n\left(\frac{(x+j)-(x+j)^p}{p}\right)-B_n\in\mathbb{Z}_p \text{ by Lemma 2.3}\right)$$

$$\equiv p^{n-1}\cdot pB_n\sum_{s=1}^{p-1}s^{b-n}$$

$$\equiv\begin{cases}0\ (\mathrm{mod}\ p^n) & \text{if } B_n\in\mathbb{Z}_p,\\ p^{n-1}(p-1)\sum_{s=1}^{p-1}s^b\equiv p^{n-1}\ (\mathrm{mod}\ p^n) & \text{if } B_n\notin\mathbb{Z}_p \text{ and } p-1\mid b,\\ p^{n-1}(p-1)\sum_{s=1}^{p-1}s^b\equiv 0\ (\mathrm{mod}\ p^n) & \text{if } B_n\notin\mathbb{Z}_p \text{ and } p-1\nmid b.\end{cases}$$

(Note that $pB_n\equiv p-1\ (\mathrm{mod}\ p)$ and that $p-1\mid n$ if $B_n\notin\mathbb{Z}_p$.)

This completes the proof. $\square$

Corollary 3.1. *Suppose that p is a prime and $x\in\mathbb{Z}_p$. If $p>n-b+1$ and $b\geqslant 0$ then*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k pB_{k(p-1)+b}(x)\equiv p^bB_b\left(\frac{x+\langle -x\rangle_p}{p}\right)+\delta(n,b,p)p^{n-1}\quad(\mathrm{mod}\ p^n),$$

where $\delta(n,b,p)$ is defined by (1.1).

Proof. Since $k(p-1)+b>n$ for $k>0$ we have

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k p^{k(p-1)+b}B_{k(p-1)+b}\left(\frac{x+\langle -x\rangle_p}{p}\right)\equiv p^bB_b\left(\frac{x+\langle -x\rangle_p}{p}\right)\quad(\mathrm{mod}\ p^n).$$

This together with Theorem 3.1 gives the result. $\square$

4. Congruences for $pB_{k(p-1)+b}(x)\ \mathrm{mod}\ p^n$

Theorem 4.1. *Let p be a prime and $x\in\mathbb{Z}_p$. For each positive integer n there are $a_0,\dots,a_n\in\mathbb{Z}_p$ such that*

$$pB_{k(p-1)+b}(x)-p^{k(p-1)+b}B_{k(p-1)+b}\left(\frac{x+\langle -x\rangle_p}{p}\right)$$
$$\equiv a_nk^n+\cdots+a_1k+a_0\ (\mathrm{mod}\ p^n)$$

for every $k=0,1,2,\dots$.

Furthermore, if $p>n$ then $a_0,a_1,\dots,a_n\ (\mathrm{mod}\ p^n)$ are uniquely determined; if $p-1\nmid b$ or $B_n\in\mathbb{Z}_p$ we may assume $a_n\equiv 0\ (\mathrm{mod}\ p^n)$.

Proof. Set

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\left(pB_{k(p-1)+b}(x)-p^{k(p-1)+b}B_{k(p-1)+b}\left(\frac{x+\langle -x\rangle_p}{p}\right)\right)=p^{n-1}A_n.$$

Then $A_0 = p\left(pB_b(x) - p^b B_b\left(\frac{x + \langle -x\rangle_p}{p}\right)\right)$. It follows from Theorem 3.1 that $A_n \in \mathbb{Z}_p$ for $n \geqslant 1$. Thus, applying binomial inversion formula we obtain

$$\begin{aligned}
&pB_{k(p-1)+b}(x) - p^{k(p-1)+b}B_{k(p-1)+b}\left(\frac{x + \langle -x\rangle_p}{p}\right)\\
&= \sum_{r=0}^{k}\binom{k}{r}(-1)^r p^{r-1}A_r \equiv \sum_{r=0}^{n}\binom{k}{r}(-1)^r p^{r-1}A_r\\
&= pB_b(x) - p^b B_b\left(\frac{x + \langle -x\rangle_p}{p}\right)\\
&\quad + \sum_{r=1}^{n} k(k-1)\cdots(k-r+1)(-1)^r \frac{p^{r-1}}{r!}A_r \pmod{p^n}.
\end{aligned} \tag{4.1}$$

Since $p^{r-1}/r! \in \mathbb{Z}_p$ for $r \geqslant 1$ we have

$$\begin{aligned}
&pB_{k(p-1)+b}(x) - p^{k(p-1)+b}B_{k(p-1)+b}\left(\frac{x + \langle -x\rangle_p}{p}\right)\\
&\equiv a_n k^n + \cdots + a_1 k + a_0 \pmod{p^n}
\end{aligned}$$

for some $a_0, a_1, \ldots, a_n \in \mathbb{Z}_p$.

Let $\{S(n,k)\}$ be the second kind Stirling numbers. From [13, 4] we know that

$$\sum_{r=0}^{m}\binom{m}{r}(-1)^{m-r}r^n = \begin{cases} m!\,S(n,m) & \text{if } n \geqslant m,\\ 0 & \text{if } n < m.\end{cases}$$

Hence,

$$\begin{aligned}
\sum_{k=0}^{m}\binom{m}{k}(-1)^k \sum_{r=0}^{n} a_r k^r &= \sum_{r=0}^{n} a_r \sum_{k=0}^{m}\binom{m}{k}(-1)^k k^r\\
&= (-1)^m m!\,(a_m + \cdots + S(n,m)a_n) \quad (m = 0, 1, \ldots, n).
\end{aligned}$$

(Note that $S(m,m) = 1$.)

For $p > n$ we have $m \leqslant n \leqslant p-1$ and so $1/m! \in \mathbb{Z}_p$. By the above it is clear that $a_n, a_{n-1}, \ldots, a_0 \pmod{p^n}$ are uniquely determined.

If $p - 1 \nmid b$ or $B_n \in \mathbb{Z}_p$, by using Theorem 3.1 we see that $A_n \equiv 0 \pmod{p}$. In view of (4.1) we get

$$\begin{aligned}
&pB_{k(p-1)+b}(x) - p^{k(p-1)+b}B_{k(p-1)+b}\left(\frac{x + \langle -x\rangle_p}{p}\right)\\
&\equiv \sum_{r=0}^{n-1}\binom{k}{r}(-1)^r p^{r-1}A_r \equiv \sum_{r=0}^{n-1} a_r k^r \pmod{p^n}
\end{aligned}$$

for some $a_0, \ldots, a_{n-1} \in \mathbb{Z}_p$.

Now, clearly we may assume $a_n \equiv 0 \pmod{p^n}$. Hence, the theorem is proved. □

Remark 4.1. Using the properties of Stirling numbers one can choose $a_0, \ldots, a_n$ so that $a_s \cdot s!/p^{s-1} \in \mathbb{Z}_p$ for every $s = 0, 1, \ldots, n$.

As examples we point out the following congruences:

$$(2 - 2^{2k})B_{2k} \equiv 12\cdot(2k)^2 + 18\cdot 2k + 1 \pmod{2^7}. \tag{4.2}$$

$$(3 - 3^{2k})B_{2k} \equiv -36k^4 + 108k^3 - 93k^2 + 18k + 2 \pmod{3^5}. \tag{4.3}$$

$$(5 - 5^{4k})B_{4k} \equiv -125k^4 + 125k^3 - 300k^2 - 100k + 4 \pmod{5^4}. \tag{4.4}$$

We remark that Frame [6] has given a congruence for $30B_{2n} \bmod 2^{11}\cdot 3^5 \cdot 5^6$. From the congruence he derived that

$$30B_{2n} \equiv 1 + 6000\binom{n-1}{2} \pmod{2^3\cdot 3^3\cdot 5^3}.$$

Corollary 4.1. *Let p be an odd prime and $x \in \mathbb{Z}_p$. If k, n and b are integers with $k, n > 0$ and $b \geqslant 0$ then*

$$pB_{k\varphi(p^n)+b}(x) \equiv pB_b(x) - p^b B_b\left(\frac{x + \langle -x\rangle_p}{p}\right) \pmod{p^n},$$

where φ denotes Euler's function.

Proof. From Theorem 3.1 we have

$$\sum_{j=0}^{1}\binom{1}{j}(-1)^j\left(pB_{j(p-1)+b}(x) - p^{j(p-1)+b}B_{j(p-1)+b}\left(\frac{x+\langle -x\rangle_p}{p}\right)\right) \equiv 0 \pmod{p}.$$

Observe that $p^{r-1}/r! \equiv 0 \pmod{p}$ for $r > 1$ and that $k\varphi(p^n) + b \geqslant p^{n-1}(p-1) > n$, by using (4.1) we obtain the result.

Remark 4.2. In [1, 2], Carlitz proved Corollary 4.1 in the case $b = 0$. When $x = 0$ and $p - 1 \nmid b$ our result can be deduced from Kummer's congruences (cf. [8, 13]). We also remark that the congruence holds for $p = 2$ if $n > 2$.

Theorem 4.2. *Let p be a prime, $k, b \in \mathbb{Z}^+ \cup \{0\}$ and $x \in \mathbb{Z}_p$. For $n = 1, 2, 3, \ldots$ we have*

$$\begin{aligned} &pB_{k(p-1)+b}(x) - p^{k(p-1)+b}B_{k(p-1)+b}\left(\frac{x+\langle -x\rangle_p}{p}\right) \\ &\equiv \sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}\Bigg(pB_{r(p-1)+b}(x) \\ &\quad - p^{r(p-1)+b}B_{r(p-1)+b}\left(\frac{x+\langle -x\rangle_p}{p}\right)\Bigg) + (-1)^n\delta(n,b,p)\binom{k}{n}p^{n-1} \pmod{p^n}, \end{aligned}$$

where

$$\delta(n,b,p)=\begin{cases}1 & \text{if } B_n\notin\mathbb{Z}_p \text{ and } p-1\mid b,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. This is immediate from Theorem 3.1 and Lemma 2.1. □

Remark 4.3. When $p-1\nmid b$ we will prove in another paper the following stronger congruences:

$$\frac{B_{k(p-1)+b}(x)-p^{k(p-1)+b-1}B_{k(p-1)+b}\left(\frac{x+\langle -x\rangle_p}{p}\right)}{k(p-1)+b}$$
$$\equiv\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}$$
$$\times\frac{B_{r(p-1)+b}(x)-p^{r(p-1)+b-1}B_{r(p-1)+b}\left(\frac{x+\langle -x\rangle_p}{p}\right)}{r(p-1)+b}$$
$$\equiv\sum_{r=0}^{n-1}a_rk^r \pmod{p^n}\ (a_0,a_1,\ldots,a_{n-1}\in\mathbb{Z}),$$

which is a wide generalization of Kummer's congruences.

Corollary 4.2. *Let p be a prime, and k, n, b be three integers with* $k\geqslant 0$, $b\geqslant 0$, $n\geqslant 1$. *Then*

$$(1-p^{k(p-1)+b-1})pB_{k(p-1)+b}\equiv\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}$$
$$\times(1-p^{r(p-1)+b-1})pB_{r(p-1)+b}+(-1)^n\delta(n,b,p)\binom{k}{n}p^{n-1}\pmod{p^n}.$$

Corollary 4.3. *Suppose* $k,n\in\mathbb{Z}^+$ *and* $x\in\mathbb{Z}_2$. *Then*

$$2B_k(x)-2^kB_k\left(\frac{x+\langle -x\rangle_2}{2}\right)\equiv\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}$$
$$\times\left(2B_r(x)-2^rB_r\left(\frac{x+\langle -x\rangle_2}{2}\right)\right)+\binom{k}{n}2^nB_n \pmod{2^n}.$$

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References

[1] L. Carlitz, A divisibility property of the Bernoulli polynomials, Proc. Amer. Math. Soc. 3 (1952) 604–607.
[2] L. Carlitz, Some congruences for the Bernoulli numbers, Amer. J. Math. 75 (1953) 163–172.
[3] L. Carlitz, Kummer's congruence for the Bernoulli numbers, Portugal. Math. 19 (1960) 203–210.
[4] D.I.A. Cohen, Basic Techniques of Combinatorial Theory (Springer, New York, 1978) 119.
[5] K. Dilcher, L. Skula and I. Sh. Slavutskii, Bernoulli Numbers Bibliography (1713–1990).
[6] J.S. Frame, Bernoulli numbers modulo 27000, Amer. Math. Monthly 68 (1961) 87–95.
[7] F.S. Gilllespie, A generalization of Kummer's congruences and related results, Fibonacci Quart. 30 (1992) 349–367.
[8] K. Ireland and M. Rosen, A Classical Introduction to Modern Number Theory (Springer, New York, 1982) 228–248.
[9] H.R. Stevens, Bernoulli numbers and Kummer's criterion, Fibonacci Quart. 24 (1986) 154–159.
[10] Z.H. Sun, A note on Wilson's theorem and Wolstenholme's theorem, Proc. Amer. Math. Soc., submitted.
[11] H.S. Vandiver, Certain congruences involving the Bernoulli numbers, Duke Math. J. 5 (1939) 548–551.
[12] H.S. Vandiver, On developments in an arithmetic theory of the Bernoulli and allied numbers, Scripta Math. 25 (1961) 273–303.
[13] L.C. Washington, Introduction to Cyclotomic Fields (Springer, New York, 1982) 61–64.