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EXPANSIONS AND IDENTITIES CONCERNING LUCAS SEQUENCES

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Abstract. In the paper we obtain some new expansions and combinatorial identities concerning Lucas sequences.

1. Introduction.

For complex numbers P and Q the Lucas sequences $\{U_n(P,Q)\}$ and $\{V_n(P,Q)\}$ are defined by

$$U_0(P,Q)=0,\ U_1(P,Q)=1,\ U_{n+1}(P,Q)=PU_n(P,Q)-QU_{n-1}(P,Q)\ (n\ge 1) \quad (1.1)$$

and

$$V_0(P,Q)=2,\ V_1(P,Q)=P,\ V_{n+1}(P,Q)=PV_n(P,Q)-QV_{n-1}(P,Q)\ (n\ge 1). \quad (1.2)$$

Set $D=P^2-4Q$. It is well known that

$$U_n(P,Q)=\begin{cases}\frac{1}{\sqrt{D}}\left\{\left(\frac{P+\sqrt{D}}{2}\right)^n-\left(\frac{P-\sqrt{D}}{2}\right)^n\right\} & \text{if } D\ne 0,\\ n\left(\frac{P}{2}\right)^{n-1} & \text{if } D=0\end{cases} \quad (1.3)$$

and

$$V_n(P,Q)=\left(\frac{P+\sqrt{D}}{2}\right)^n+\left(\frac{P-\sqrt{D}}{2}\right)^n. \quad (1.4)$$

In Section 2 we state various expansions for $U_n(P,Q)$ and illustrate the connections among them. In Section 3 we investigate the properties of $\{S_n(x)\}$ and $\{G_n(x)\}$, where

$$S_n(x)=\sum_{k=0}^{n}\frac{2n+1}{2n+1-k}\binom{2n+1-k}{k}x^{n-k} \quad \text{and} \quad G_n(x)=\sum_{k=0}^{n}(-1)^{[\frac{n-k}{2}]}\binom{[\frac{n+k}{2}]}{k}x^k.$$

For example, we have $S_n(x)=G_n(x+2)$. Let $U_n=U_n(P,Q)$ and $V_n=V_n(P,Q)$. In Section 3 we also establish the following identity:

$$U_{(2n+1)k}=U_k\sum_{m=0}^{n}(-1)^{[\frac{n-m}{2}]}\binom{[\frac{n+m}{2}]}{m}Q^{k(n-m)}V_{2k}^m,$$

where $[x]$ denotes the greatest integer not exceeding x.

In Section 4, using the results in Sections 2 and 3 we establish several combinatorial identities. For example, if m and n are nonnegative integers with $m \le n$, then

$$\frac{2n+1}{2m+1}\binom{n+m}{2m} = \sum_{k=m}^{n}(-1)^{[\frac{n-k}{2}]}\binom{[\frac{n+k}{2}]}{k}\binom{k}{m}2^{k-m} = \sum_{k=m}^{n}(-1)^{n-k}\binom{n+k}{2k}\binom{k}{m}4^{k-m}$$
$$= \frac{1}{4^m}\sum_{k=0}^{m}\binom{2n+1}{2k+1}\binom{n-k}{n-m}.$$

2. Expansions for $U_n(P,Q)$.

Let $U_n = U_n(P,Q)$ and $V_n = V_n(P,Q)$ be the Lucas sequences given by (1.1) and (1.2). From (1.3) and (1.4) one can easily check the following known facts (cf.[1,4,5,8]):

$$V_n = PU_n - 2QU_{n-1} = 2U_{n+1} - PU_n = U_{n+1} - QU_{n-1}, \tag{2.1}$$
$$U_{2n} = U_nV_n,\ V_{2n} = V_n^2 - 2Q^n, \tag{2.2}$$
$$V_n^2 - (P^2-4Q)U_n^2 = 4Q^n, \tag{2.3}$$
$$U_{n+k} = V_kU_n - Q^kU_{n-k}\ (n \ge k). \tag{2.4}$$

By (2.4), if $U_k \ne 0$, then $U_{k(n+1)}/U_k = V_kU_{kn}/U_k - Q^kU_{k(n-1)}/U_k$. Thus

$$U_{kn}/U_k = U_n(V_k, Q^k). \tag{2.5}$$

Since $U_2 = P$ and $V_2 = P^2 - 2Q$, by (2.5) we have

$$U_{2n}(P,Q) = PU_n(P^2-2Q, Q^2). \tag{2.6}$$

Next we look at certain expansions for $U_n(P,Q)$. By induction one can prove the following well known result (cf. [5, (2.5)], [2, (1.60), (1.61), (1.64)], [7, Lemma 1.4] and [8, (4.2.36)])

$$U_{n+1}(P,Q) = \sum_{k=0}^{[\frac{n}{2}]}\binom{n-k}{k}(-Q)^kP^{n-2k} \tag{2.7}$$

and

$$V_n(P,Q) = \sum_{k=0}^{[n/2]}\frac{n}{n-k}\binom{n-k}{k}P^{n-2k}(-Q)^k. \tag{2.8}$$

Combining (2.6) and (2.7) we get

$$U_{2n+2}(P,Q) = P\sum_{k=0}^{[\frac{n}{2}]}\binom{n-k}{k}(-Q^2)^k(P^2-2Q)^{n-2k}. \tag{2.9}$$

From (1.3) and the binomial theorem one can easily deduce another expansion:

$$U_{n+1}(P,Q)=\frac{1}{2^n}\sum_{k=0}^{[\frac{n}{2}]}\binom{n+1}{2k+1}P^{n-2k}(P^2-4Q)^k. \tag{2.10}$$

This together with (2.6) gives

$$U_{2n+2}(P,Q)=\frac{P}{2^n}\sum_{k=0}^{[\frac{n}{2}]}\binom{n+1}{2k+1}(P^2(P^2-4Q))^k(P^2-2Q)^{n-2k}. \tag{2.11}$$

Using (1.3) and (1.4) one can easily prove the following transformation formulas:

$$U_{2n}(P,Q)=\frac{P}{\sqrt{P^2-4Q}}U_{2n}(\sqrt{P^2-4Q},-Q), \tag{2.12}$$

$$V_{2n}(P,Q)=V_{2n}(\sqrt{P^2-4Q},-Q), \tag{2.13}$$

$$U_{2n+1}(P,Q)=\frac{1}{\sqrt{P^2-4Q}}V_{2n+1}(\sqrt{P^2-4Q},-Q), \tag{2.14}$$

$$V_{2n+1}(P,Q)=PU_{2n+1}(\sqrt{P^2-4Q},-Q). \tag{2.15}$$

Here (2.12)-(2.15) are due to my twin brother Zhi-Wei Sun (he never published these formulas). From (2.12) and (2.7) we see that

$$U_{2n+2}(P,Q)=P\sum_{k=0}^{n}\binom{2n+1-k}{k}Q^k(P^2-4Q)^{n-k}. \tag{2.16}$$

Combining (2.14) with (2.8) yields

$$U_{2n+1}(P,Q)=\sum_{k=0}^{n}\frac{2n+1}{2n+1-k}\binom{2n+1-k}{k}Q^k(P^2-4Q)^{n-k}. \tag{2.17}$$

Thus, if $U_m=U_m(P,Q)$ and $U_k\neq 0$, applying (2.5), (2.3) and (2.17) we have

$$\frac{U_{(2n+1)k}}{U_k}=\sum_{m=0}^{n}\frac{2n+1}{2n+1-m}\binom{2n+1-m}{m}Q^{km}\big((P^2-4Q)U_k^2\big)^{n-m}. \tag{2.18}$$

3. The polynomials $S_n(x)$ and $G_n(x)$.

For any positive integer n and $k\in\{0,1,\dots,[n/2]\}$ define

$$C_{n,k}=\frac{n}{n-k}\binom{n-k}{k}.$$

It is clear that

$$C_{n,k} = \binom{n-k}{k} + \binom{n-1-k}{k-1} = \frac{n}{k}\binom{n-1-k}{k-1}$$
$$= \frac{n}{n-2k}\binom{n-1-k}{k} = \frac{n\cdot(n-1-k)!}{k!(n-2k)!}.$$

By (2.8) we have

$$V_n(x,a) = \sum_{k=0}^{[n/2]} C_{n,k}(-a)^k x^{n-2k}.$$

$C_{n,k}$ also concerns with the first Chebyshev polynomial $T_n(x)$ (in fact, $V_n(x,1) = 2T_n(x/2)$) and Dickson polynomial $D_n(x,a)$ (in fact, $D_n(x,a) = V_n(x,a)$). See also [6].

Definition 3.1. *For nonnegative integer n and complex number x define*

$$S_n(x) = \sum_{k=0}^{n} \frac{2n+1}{2n+1-k}\binom{2n+1-k}{k} x^{n-k} = \sum_{k=0}^{n} C_{2n+1,k}x^{n-k}.$$

The first few $S_n(x)$ are shown below:

$$S_0(x) = 1,\ S_1(x) = x+3,\ S_2(x) = x^2+5x+5,$$
$$S_3(x) = x^3+7x^2+14x+7,\ S_4(x) = x^4+9x^3+27x^2+30x+9,$$
$$S_5(x) = x^5+11x^4+44x^3+77x^2+55x+11.$$

Theorem 3.1. *$\{S_n(x)\}$ is given by $S_0(x) = 1$, $S_1(x) = x+3$ and $S_{n+1}(x) = (x+2)S_n(x) - S_{n-1}(x)$ $(n \ge 1)$.*

Proof. By (2.17) we have $U_{2n+1}(\sqrt{x+4},1) = S_n(x)$. Taking $k = 2$ in (2.4) we find

$$U_{2n+3}(P,Q) = (P^2-2Q)U_{2n+1}(P,Q) - Q^2U_{2n-1}(P,Q).$$

Thus, for $n \ge 1$,

$$S_{n+1}(x) = U_{2n+3}(\sqrt{x+4},1) = (x+2)U_{2n+1}(\sqrt{x+4},1) - U_{2n-1}(\sqrt{x+4},1)$$
$$= (x+2)S_n(x) - S_{n-1}(x).$$

This together with the fact that $S_0(x) = 1$ and $S_1(x) = x+3$ proves the theorem.

In [7] the author introduced

$$G_n(x) = \prod_{r=1}^{n}\left(x + 2\cos\frac{2r-1}{2n+1}\pi\right)$$

and showed that

$$G_0(x) = 1,\ G_1(x) = x+1,\quad G_{n+1}(x) = xG_n(x) - G_{n-1}(x)\ (n \ge 1) \tag{3.1}$$

and

$$G_n(x) = \sum_{k=0}^{n}(-1)^{[\frac{n-k}{2}]}\binom{[\frac{n+k}{2}]}{k}x^k = U_n(x,1) + U_{n+1}(x,1). \tag{3.2}$$

Theorem 3.2. *For nonnegative integer n and nonzero complex number x we have*

$$\begin{aligned} S_n(x) &= G_n(x+2) = \frac{1}{\sqrt{x}} V_{2n+1}(\sqrt{x},-1) = U_{2n+1}(\sqrt{x+4},1) \\ &= U_n(x+2,1) + U_{n+1}(x+2,1). \end{aligned}$$

Proof. The result follows from (2.8), (2.17), (3.1), (3.2) and Theorem 3.1.

Theorem 3.3. *For complex numbers P, Q $(Q \ne 0)$ and nonnegative integer n we have*

$$U_{2n+1}(P,Q) = Q^n G_n\Big(\frac{P^2-2Q}{Q}\Big) = \sum_{k=0}^{n} (-1)^{[\frac{n-k}{2}]} \binom{[\frac{n+k}{2}]}{k} Q^{n-k}(P^2-2Q)^k.$$

Proof. From (2.17) and Theorem 3.2 we see that

$$U_{2n+1}(P,Q) = Q^n S_n\Big(\frac{P^2-4Q}{Q}\Big) = Q^n G_n\Big(\frac{P^2-4Q}{Q}+2\Big) = Q^n G_n\Big(\frac{P^2-2Q}{Q}\Big).$$

Thus applying (3.2) we obtain the result.

Let $\mathbb{Z}$ be the set of integers. From Theorem 3.3 we have

Theorem 3.4. *If $n, k \in \mathbb{Z}$, $n \ge 0, k \ge 1$, $U_m = U_m(P,Q)$, $V_m = V_m(P,Q)$ and $QU_k \ne 0$, then*

$$\frac{U_{(2n+1)k}}{U_k} = \sum_{m=0}^{n} (-1)^{[\frac{n-m}{2}]} \binom{[\frac{n+m}{2}]}{m} Q^{k(n-m)} V_{2k}^m.$$

Proof. From (2.5) we know that $U_{(2n+1)k}/U_k = U_{2n+1}(P',Q')$, where $P' = V_k$ and $Q' = Q^k$. Since ${P'}^2 - 2Q' = V_{2k}$ by (2.2), applying Theorem 3.3 we obtain the result.

4. Some related combinatorical identities.

Putting $P = 1$ and $Q = -x$ in (2.7), (2.9), (2.10), (2.11), (2.16) and then comparing the expansions for $U_{2n+2}(1,-x)$ we obtain the following result.

Theorem 4.1. *Let n be a nonnegative integer, and let x be a complex number. Then*

$$\begin{aligned} \sum_{k=0}^{n} \binom{2n+1-k}{k} x^k &= \sum_{k=0}^{[\frac{n}{2}]} \binom{n-k}{k} (-1)^k x^{2k} (1+2x)^{n-2k} \\ &= \frac{1}{2^{2n+1}} \sum_{k=0}^{n} \binom{2n+2}{2k+1} (1+4x)^k \\ &= \frac{1}{2^n} \sum_{k=0}^{[\frac{n}{2}]} \binom{n+1}{2k+1} (1+4x)^k (1+2x)^{n-2k} \\ &= \sum_{k=0}^{n} \binom{2n+1-k}{k} (-x)^k (1+4x)^{n-k}. \end{aligned}$$

By comparing the coefficients of x^m in Theorem 4.1 we have

Theorem 4.2. *Let n and m be two integers with $0 \le m \le n$. Then*

$$\begin{aligned}\binom{2n+1-m}{m} &= \sum_{k=0}^{[\frac{m}{2}]} \binom{n-k}{k}(-1)^k \binom{n-2k}{n-m} 2^{m-2k} \\ &= 2^{2m-2n-1} \sum_{k=m}^{n} \binom{2n+2}{2k+1}\binom{k}{m} \\ &= \sum_{k=0}^{m} \binom{2n+1-k}{k}(-1)^k \binom{n-k}{n-m} 4^{m-k}.\end{aligned}$$

Theorem 4.3. *For any nonnegative integer n and complex number x,*

$$\begin{aligned}\sum_{k=0}^{n} \frac{2n+1}{2k+1}\binom{n+k}{2k} x^k &= \sum_{k=0}^{n} (-1)^{[\frac{n-k}{2}]} \binom{[\frac{n+k}{2}]}{k} (x+2)^k \\ &= \sum_{k=0}^{n} \binom{2n-k}{k}(-1)^k (x+4)^{n-k} \\ &= \frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{2n+1}{2k+1} x^k (x+4)^{n-k}.\end{aligned}$$

Proof. Clearly

$$\sum_{k=0}^{n} \frac{2n+1}{2k+1}\binom{n+k}{2k} x^k = \sum_{k=0}^{n} \frac{2n+1}{2n+1-2k}\binom{2n-k}{k} x^{n-k} = S_n(x).$$

Since $S_n(x) = G_n(x+2)$, by (3.2) we have

$$S_n(x) = \sum_{k=0}^{n} (-1)^{[\frac{n-k}{2}]} \binom{[\frac{n+k}{2}]}{k} (x+2)^k.$$

On the other hand, by Theorem 3.2, $S_n(x) = U_{2n+1}(\sqrt{x+4}, 1)$. Applying (2.7) and (2.10) we get

$$S_n(x) = \sum_{k=0}^{n} \binom{2n-k}{k}(-1)^k (x+4)^{n-k} = \frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{2n+1}{2k+1} x^k (x+4)^{n-k}.$$

Combining the above proves the theorem.

Theorem 4.4. *If m and n are two nonnegative integers with $m \le n$, then*

$$\begin{aligned}\frac{2n+1}{2m+1}\binom{n+m}{2m} &= \binom{n+m}{2m+1} + \binom{n+m+1}{2m+1} = \sum_{k=m}^{n} (-1)^{[\frac{n-k}{2}]} \binom{[\frac{n+k}{2}]}{k}\binom{k}{m} 2^{k-m} \\ &= \sum_{k=m}^{n} (-1)^{n-k} \binom{n+k}{2k}\binom{k}{m} 4^{k-m} = \frac{1}{4^m} \sum_{k=0}^{m} \binom{2n+1}{2k+1}\binom{n-k}{n-m}.\end{aligned}$$

Proof. It's easy to verify that

$$\frac{2n+1}{2m+1}\binom{n+m}{2m} = \binom{n+m}{2m+1} + \binom{n+m+1}{2m+1}.$$

Since

$$\begin{aligned}\sum_{k=0}^{n}(-1)^{[\frac{n-k}{2}]}\binom{[\frac{n+k}{2}]}{k}(x+2)^k &= \sum_{k=0}^{n}(-1)^{[\frac{n-k}{2}]}\binom{[\frac{n+k}{2}]}{k}\sum_{m=0}^{k}\binom{k}{m}2^{k-m}x^m \\ &= \sum_{m=0}^{n}\sum_{k=m}^{n}(-1)^{[\frac{n-k}{2}]}\binom{[\frac{n+k}{2}]}{k}\binom{k}{m}2^{k-m}x^m,\end{aligned}$$

$$\begin{aligned}\sum_{k=0}^{n}\binom{2n-k}{k}(-1)^k(x+4)^{n-k} &= \sum_{k=0}^{n}\binom{2n-k}{k}(-1)^k\sum_{m=0}^{n-k}\binom{n-k}{m}4^{n-k-m}x^m \\ &= \sum_{m=0}^{n}\sum_{k=0}^{n-m}\binom{2n-k}{k}\binom{n-k}{m}(-1)^k 4^{n-k-m}x^m \\ &= \sum_{m=0}^{n}\sum_{k=m}^{n}\binom{n+k}{2k}\binom{k}{m}(-1)^{n-k}4^{k-m}x^m\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2^{2n}}\sum_{k=0}^{n}\binom{2n+1}{2k+1}x^k(x+4)^{n-k} &= \frac{1}{2^{2n}}\sum_{k=0}^{n}\binom{2n+1}{2k+1}\sum_{m=0}^{n-k}\binom{n-k}{m}4^m x^{n-m} \\ &= \frac{1}{2^{2n}}\sum_{k=0}^{n}\binom{2n+1}{2k+1}\sum_{m=k}^{n}\binom{n-k}{n-m}4^{n-m}x^m \\ &= \sum_{m=0}^{n}\frac{1}{4^m}\sum_{k=0}^{m}\binom{2n+1}{2k+1}\binom{n-k}{n-m}x^m,\end{aligned}$$

by comparing the coefficients of x^m in Theorem 4.3 we obtain the result.

Theorem 4.5. *For any nonnegative integer n,*

$$\sum_{k=0}^{n}(-1)^{[\frac{n-k}{2}]}\binom{[\frac{n+k}{2}]}{k} = \begin{cases}(-1)^n & \text{if } n \not\equiv 1 \pmod 3, \\ 2(-1)^{n+1} & \text{if } n \equiv 1 \pmod 3\end{cases}$$

and

$$\sum_{k=0}^{n}(-1)^{[\frac{n-k}{2}]}\binom{[\frac{n+k}{2}]}{k}3^k = L_{2n+1},$$

where $L_m = V_m(1,-1)$ is the Lucas sequence.

Proof. From Theorem 3.2 we see that $G_n(3) = L_{2n+1}$ and

$$\begin{aligned} G_n(1) = U_{2n+1}(\sqrt{3},1) &= \frac{1}{\sqrt{-1}}\Big\{\Big(\frac{\sqrt{3}+\sqrt{-1}}{2}\Big)^{2n+1} - \Big(\frac{\sqrt{3}-\sqrt{-1}}{2}\Big)^{2n+1}\Big\} \\ &= \frac{1}{(\sqrt{-1})^{2n+2}}\Big\{\Big(\frac{-1+\sqrt{-3}}{2}\Big)^{2n+1} + \Big(\frac{-1-\sqrt{-3}}{2}\Big)^{2n+1}\Big\} \\ &= (-1)^{n+1}(\omega^{2n+1}+\omega^{2(2n+1)}) = \begin{cases} (-1)^n & \text{if } n \not\equiv 1 \pmod 3, \\ 2(-1)^{n+1} & \text{if } n \equiv 1 \pmod 3, \end{cases} \end{aligned}$$

where $\omega = (-1+\sqrt{-3})/2$. Thus applying (3.2) yields the result.
Remark 4.1 By (2.8) and (1.4),

$$\begin{aligned} \sum_{k=0}^{[n/2]}(-1)^{n-k}\frac{n}{n-k}\binom{n-k}{k} &= V_n(-1,1) = \Big(\frac{-1+\sqrt{-3}}{2}\Big)^n + \Big(\frac{-1-\sqrt{-3}}{2}\Big)^n \\ &= \omega^n + \omega^{2n} = \begin{cases} 2 & \text{if } 3 \mid n, \\ -1 & \text{if } 3 \nmid n. \end{cases} \end{aligned}$$

See [3, Exercise 44, p.445] and [2, (1.68)].

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