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CONGRUENCES INVOLVING BERNOULLI POLYNOMIALS

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Abstract. Let $\{B_n(x)\}$ be the Bernoulli polynomials. In the paper we establish some congruences for $B_j(x) \pmod{p^n}$, where p is an odd prime and x is a rational p-integer. Such congruences are concerned with the properties of p-regular functions, the congruences for $h(-sp) \pmod p$ $(s = 3, 5, 8, 12)$ and the sum $\sum_{k\equiv r \pmod m} \binom{p}{k}$, where $h(d)$ is the class number of the quadratic field $\mathbb{Q}(\sqrt{d})$ of discriminant d and p-regular functions are those functions f such that $f(k)$ $(k = 0, 1, \dots)$ are rational p-integers and $\sum_{k=0}^{n} \binom{n}{k}(-1)^k f(k) \equiv 0 \pmod{p^n}$ for $n = 1, 2, 3, \dots$ We also establish many congruences for Euler numbers.

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1. Introduction.

The Bernoulli numbers $\{B_n\}$ and Bernoulli polynomials $\{B_n(x)\}$ are defined by

$$B_0 = 1,\ \sum_{k=0}^{n-1} \binom{n}{k} B_k = 0\ (n \ge 2) \quad \text{and} \quad B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}\ (n \ge 0).$$

The Euler numbers $\{E_n\}$ and Euler polynomials $\{E_n(x)\}$ are defined by

$$\frac{2\mathrm{e}^t}{\mathrm{e}^{2t}+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}\ (|t| < \frac{\pi}{2}) \quad \text{and} \quad \frac{2\mathrm{e}^{xt}}{\mathrm{e}^t+1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}\ (|t| < \pi),$$

which are equivalent to (see [MOS])

$$E_0 = 1,\ E_{2n-1} = 0,\ \sum_{r=0}^{n} \binom{2n}{2r} E_{2r} = 0\ (n \ge 1)$$

and

$$(1.1) \qquad E_n(x) + \sum_{r=0}^{n} \binom{n}{r} E_r(x) = 2x^n\ (n \ge 0).$$

It is well known that([MOS])

$$(1.2)\qquad \begin{aligned} E_n(x) &= \frac{1}{2^n}\sum_{r=0}^{n}\binom{n}{r}(2x-1)^{n-r}E_r \\ &= \frac{2}{n+1}\Big(B_{n+1}(x)-2^{n+1}B_{n+1}\Big(\frac{x}{2}\Big)\Big). \end{aligned}$$

Let $\mathbb{Z}$ and $\mathbb{N}$ be the set of integers and the set of positive integers respectively. Let $[x]$ be the integral part of x and $\{x\}$ be the fractional part of x. If $m,s\in\mathbb{N}$ and p is an odd prime not dividing m, in Section 2 we show that

$$(-1)^s\frac{m}{p}\sum_{\substack{k=1\\ k\equiv sp\pmod m}}^{p-1}\binom{p}{k}$$
$$\equiv\begin{cases} B_{p-1}\big(\{\frac{(s-1)p}{m}\}\big)-B_{p-1}\big(\{\frac{sp}{m}\}\big)\pmod p & \text{if } 2\mid m,\\ \frac{1}{2}\big((-1)^{[\frac{(s-1)p}{m}]}E_{p-2}\big(\{\frac{(s-1)p}{m}\}\big)-(-1)^{[\frac{sp}{m}]}E_{p-2}\big(\{\frac{sp}{m}\}\big)\big)\pmod p & \text{if } 2\nmid m.\end{cases}$$

For a discriminant d let $h(d)$ be the class number of the quadratic field $\mathbb{Q}(\sqrt{d})$ ($\mathbb{Q}$ is the set of rational numbers). If $p>3$ is a prime of the form $4m+3$, it is well known that (cf. [IR])

$$(1.3)\qquad h(-p)\equiv -2B_{\frac{p+1}{2}}\pmod p.$$

If p is a prime of the form $4m+1$, according to [Er] we have

$$(1.4)\qquad 2h(-4p)\equiv E_{\frac{p-1}{2}}\pmod p.$$

Let $(\frac{a}{n})$ be the Kronecker symbol. For odd primes p, in Section 3 we establish the following congruences:

$$\begin{aligned} &h(-8p)\equiv E_{\frac{p-1}{2}}\Big(\frac{1}{4}\Big)\pmod p;\\ &h(-3p)\equiv -4\Big(\frac{p}{3}\Big)B_{\frac{p+1}{2}}\Big(\frac{1}{3}\Big)\pmod p\quad\text{for } p\equiv 1\pmod 4;\\ &h(-12p)\equiv 8\Big(\frac{p}{3}\Big)B_{\frac{p+1}{2}}\Big(\frac{1}{12}\Big)\pmod p\quad\text{for } p\equiv 7,11,23\pmod{24};\\ &h(-5p)\equiv -8B_{\frac{p+1}{2}}\Big(\frac{1}{5}\Big)\pmod p\quad\text{for } p\equiv 11,19\pmod{20}. \end{aligned}$$

For $m\in\mathbb{N}$ let $\mathbb{Z}_m$ be the set of rational numbers whose denominator is coprime to m. For a prime p, in [S5] the author introduced the notion of p-regular functions. If $f(k)\in\mathbb{Z}_p$ for any nonnegative integers k and $\sum_{k=0}^{n}\binom{n}{k}(-1)^k f(k)\equiv 0\pmod{p^n}$

for all $n \in \mathbb{N}$, then f is called a p-regular function. If f is a p-regular function and $k, m, n, t \in \mathbb{N}$, in Section 4 we show that

$$f(ktp^{m-1}) \equiv \sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}f(rtp^{m-1}) \pmod{p^{mn}}, \tag{1.5}$$

which was annouced by the author in [S5, (2.4)]. We also show that

$$f(kp^{m-1}) \equiv (1-kp^{m-1})f(0)+kp^{m-1}f(1) \pmod{p^{m+1}} \quad \text{for} \quad p>2. \tag{1.6}$$

Let p be a prime, $x \in \mathbb{Z}_p$ and let b be a nonnegative integer. Let $\langle t\rangle_p$ be the least nonnegative residue of t modulo p and $x' = (x+\langle -x\rangle_p)/p$. From [S4, Theorem 3.1] we know that $f(k) = p(pB_{k(p-1)+b}(x) - p^{k(p-1)+b}B_{k(p-1)+b}(x'))$ is a $p-$regular function. If $p-1 \nmid b$, in [S5] the author showed that $f(k) = (B_{k(p-1)+b}(x) - p^{k(p-1)+b-1}B_{k(p-1)+b}(x'))/(k(p-1)+b)$ is also a $p-$regular function. Using such results in [S4, S5] and (1.5), in Section 5 we obtain general congruences for $pB_{k\varphi(p^s)+b}(x)$, $pB_{k\varphi(p^s)+b,\chi} \pmod{p^{sn}}$, where $k, n, s \in \mathbb{N}$, φ is Euler's totient function and χ is a Dirichlet character modulo a positive integer. As a consequence of (1.6), if $2 \mid b$ and $p-1 \nmid b$, we have

$$\frac{B_{k\varphi(p^s)+b}}{k\varphi(p^s)+b} \equiv (1-kp^{s-1})(1-p^{b-1})\frac{B_b}{b} + kp^{s-1}\frac{B_{p-1+b}}{p-1+b} \pmod{p^{s+1}}.$$

In Section 6 we establish some congruences for $\sum_{k=0}^{n}\binom{n}{k}(-1)^k pB_{k(p-1)+b}(x)$ modulo p^{n+1}, where p is an odd prime, $n \in \mathbb{N}$, $x \in \mathbb{Z}_p$ and b is a nonnegative integer.

Let p be an odd prime and $b \in \{0, 2, 4, \dots\}$. In Section 7 we show that $f(k) = (1-(-1)^{\frac{p-1}{2}}p^{k(p-1)+b})E_{k(p-1)+b}$ is a $p-$regular function. Using this and (1.5) we give congruences for $E_{k\varphi(p^m)+b} \pmod{p^{mn}}$, where $k, m \in \mathbb{N}$. By (1.6) we have

$$E_{k\varphi(p^m)+b} \equiv (1-kp^{m-1})(1-(-1)^{\frac{p-1}{2}}p^b)E_b + kp^{m-1}E_{p-1+b} \pmod{p^{m+1}}.$$

We also show that $f(k) = E_{2k+b}$ is a $2-$regular function and

$$E_{2^m kt+b} \equiv \sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}E_{2^m rt+b} \pmod{2^{mn+n-\alpha}},$$

where $k, m, n, t \in \mathbb{N}$ and $\alpha \in \mathbb{N}$ is given by $2^{\alpha-1} \le n < 2^{\alpha}$.

2. Congruences for $B_k(\{\frac{(s-1)p}{m}\}) - B_k(\{\frac{sp}{m}\}) \pmod{p}$.

We begin with two useful identities concerning Bernoulli and Euler polynomials. In the case $m = 1$ the result is well known. See [MOS].

Theorem 2.1. *Let $p, m \in \mathbb{N}$ and $k, r \in \mathbb{Z}$ with $k \geq 0$. Then*

$$\sum_{\substack{x=0 \\ x\equiv r(\text{mod } m)}}^{p-1} x^k = \frac{m^k}{k+1}\Big(B_{k+1}\Big(\frac{p}{m}+\Big\{\frac{r-p}{m}\Big\}\Big) - B_{k+1}\Big(\Big\{\frac{r}{m}\Big\}\Big)\Big)$$

and

$$\sum_{\substack{x=0 \\ x\equiv r(\text{mod } m)}}^{p-1} (-1)^{\frac{x-r}{m}} x^k = -\frac{m^k}{2}\Big((-1)^{[\frac{r-p}{m}]}E_k\Big(\frac{p}{m}+\Big\{\frac{r-p}{m}\Big\}\Big) - (-1)^{[\frac{r}{m}]}E_k\Big(\Big\{\frac{r}{m}\Big\}\Big)\Big).$$

Proof. For any real number t and nonnegative integer n it is well known that (cf. [MOS])

$$\text{(2.1)} \qquad B_n(t+1) - B_n(t) = nt^{n-1} \ (n \neq 0) \quad \text{and} \quad E_n(t+1) + E_n(t) = 2t^n.$$

Hence, for $x \in \mathbb{Z}$ we have

$$\begin{aligned} &B_{k+1}\Big(\frac{x+1}{m}+\Big\{\frac{r-x-1}{m}\Big\}\Big) - B_{k+1}\Big(\frac{x}{m}+\Big\{\frac{r-x}{m}\Big\}\Big) \\ &= \begin{cases} B_{k+1}\big(\frac{x+1}{m}+\{\frac{r-x}{m}\}-\frac{1}{m}\big) - B_{k+1}\big(\frac{x}{m}+\{\frac{r-x}{m}\}\big) = 0 & \text{if } m \nmid x-r, \\ B_{k+1}\big(\frac{x+1}{m}+\frac{m-1}{m}\big) - B_{k+1}\big(\frac{x}{m}\big) = (k+1)\big(\frac{x}{m}\big)^k & \text{if } m \mid x-r. \end{cases} \end{aligned}$$

Thus

$$\begin{aligned} &B_{k+1}\Big(\frac{p}{m}+\Big\{\frac{r-p}{m}\Big\}\Big) - B_{k+1}\Big(\Big\{\frac{r}{m}\Big\}\Big) \\ &= \sum_{x=0}^{p-1}\Big(B_{k+1}\Big(\frac{x+1}{m}+\Big\{\frac{r-x-1}{m}\Big\}\Big) - B_{k+1}\Big(\frac{x}{m}+\Big\{\frac{r-x}{m}\Big\}\Big)\Big) \\ &= \frac{k+1}{m^k}\sum_{\substack{x=0 \\ x\equiv r(\text{mod } m)}}^{p-1} x^k. \end{aligned}$$

Similarly, if $x \in \mathbb{Z}$, by (2.1) we have

$$\begin{aligned} &(-1)^{[\frac{r-x-1}{m}]}E_k\Big(\frac{x+1}{m}+\Big\{\frac{r-x-1}{m}\Big\}\Big) - (-1)^{[\frac{r-x}{m}]}E_k\Big(\frac{x}{m}+\Big\{\frac{r-x}{m}\Big\}\Big) \\ &= \begin{cases} (-1)^{[\frac{r-x}{m}]}\big(E_k\big(\frac{x+1}{m}+\{\frac{r-x}{m}\}-\frac{1}{m}\big) - E_k\big(\frac{x}{m}+\{\frac{r-x}{m}\}\big)\big) = 0 \\ \qquad\qquad\qquad\qquad \text{if } m \nmid x-r, \\ (-1)^{\frac{r-x}{m}-1}E_k\big(\frac{x+1}{m}+\frac{m-1}{m}\big) - (-1)^{\frac{r-x}{m}}E_k\big(\frac{x}{m}\big) = -(-1)^{\frac{r-x}{m}}\cdot 2\big(\frac{x}{m}\big)^k \\ \qquad\qquad\qquad\qquad \text{if } m \mid x-r. \end{cases} \end{aligned}$$

Thus

$$\begin{aligned}
&(-1)^{[\frac{r-p}{m}]}E_k\Big(\frac{p}{m}+\Big\{\frac{r-p}{m}\Big\}\Big)-(-1)^{[\frac{r}{m}]}E_k\Big(\Big\{\frac{r}{m}\Big\}\Big)\\
&=\sum_{x=0}^{p-1}\Big\{(-1)^{[\frac{r-x-1}{m}]}E_k\Big(\frac{x+1}{m}+\Big\{\frac{r-x-1}{m}\Big\}\Big)\\
&\qquad -(-1)^{[\frac{r-x}{m}]}E_k\Big(\frac{x}{m}+\Big\{\frac{r-x}{m}\Big\}\Big)\Big\}\\
&=-\frac{2}{m^k}\sum_{\substack{x=0\\ x\equiv r \pmod m}}^{p-1}(-1)^{\frac{x-r}{m}}x^k.
\end{aligned}$$

This completes the proof.

Corollary 2.1. *Let p be an odd prime and $k\in\{0,1,\dots,p-2\}$. Let $r\in\mathbb{Z}$ and $m\in\mathbb{N}$ with $p\nmid m$. Then*

$$\sum_{\substack{x=0\\ x\equiv r \pmod m}}^{p-1}x^k\equiv\frac{m^k}{k+1}\Big(B_{k+1}\Big(\Big\{\frac{r-p}{m}\Big\}\Big)-B_{k+1}\Big(\Big\{\frac{r}{m}\Big\}\Big)\Big)\pmod p$$

and

$$\begin{aligned}
&\sum_{\substack{x=0\\ x\equiv r \pmod m}}^{p-1}(-1)^{\frac{x-r}{m}}x^k\\
&\equiv-\frac{m^k}{2}\Big((-1)^{[\frac{r-p}{m}]}E_k\Big(\Big\{\frac{r-p}{m}\Big\}\Big)-(-1)^{[\frac{r}{m}]}E_k\Big(\Big\{\frac{r}{m}\Big\}\Big)\Big)\pmod p.
\end{aligned}$$

Proof. If $x_1,x_2\in\mathbb{Z}_p$ and $x_1\equiv x_2\pmod p$, by [S5, Lemma 3.1] and [S3, Lemma 3.3] we have

$$\frac{B_{k+1}(x_1)-B_{k+1}(x_2)}{k+1}\equiv\frac{x_1-x_2}{p}\cdot pB_k\equiv 0\pmod p \tag{2.2}$$

and

$$E_k(x_1)\equiv E_k(x_2)\pmod p. \tag{2.3}$$

Thus the result follows from Theorem 2.1.

Remark 2.1 Putting $k=p-2$ in Corollary 2.1 and then applying Fermat's little theorem we see that if p is an odd prime not dividing m, then

$$\sum_{\substack{x=1\\ x\equiv r \pmod m}}^{p-1}\frac{1}{x}\equiv-\frac{1}{m}\Big(B_{p-1}\Big(\Big\{\frac{r-p}{m}\Big\}\Big)-B_{p-1}\Big(\Big\{\frac{r}{m}\Big\}\Big)\Big)\pmod p \tag{2.4}$$

and

$$\sum_{\substack{x=1\\x\equiv r(\text{mod } m)}}^{p-1}(-1)^{\frac{x-r}{m}}\frac{1}{x} \tag{2.5}$$
$$\equiv -\frac{1}{2m}\Big((-1)^{[\frac{r-p}{m}]}E_{p-2}\Big(\Big\{\frac{r-p}{m}\Big\}\Big)-(-1)^{[\frac{r}{m}]}E_{p-2}\Big(\Big\{\frac{r}{m}\Big\}\Big)\Big) \pmod p.$$

Here (2.4) and (2.5) are due to my brother Z.W. Sun. See [Su2, Theorem 2.1]. Inspired by his work, the author established Theorem 2.1 and Corollary 2.1 in 1991.

Corollary 2.2. *Let p be an odd prime. Let $k\in\{0,1,\dots,p-2\}$ and $m,s\in\mathbb{N}$ with $p\nmid m$. Then*

$$\frac{(-1)^k}{k+1}\Big(B_{k+1}\Big(\Big\{\frac{(s-1)p}{m}\Big\}\Big)-B_{k+1}\Big(\Big\{\frac{sp}{m}\Big\}\Big)\Big)\equiv\sum_{\frac{(s-1)p}{m}<r\le\frac{sp}{m}}r^k \pmod p$$

and

$$(-1)^{[\frac{(s-1)p}{m}]}E_k\Big(\Big\{\frac{(s-1)p}{m}\Big\}\Big)-(-1)^{[\frac{sp}{m}]}E_k\Big(\Big\{\frac{sp}{m}\Big\}\Big)$$
$$\equiv 2(-1)^{k-1}\sum_{\frac{(s-1)p}{m}<r\le\frac{sp}{m}}(-1)^r r^k \pmod p.$$

Proof. It is clear that (see [S3, Lemma 3.1, Corollaries 3.1 and 3.3])

$$\sum_{\substack{x=0\\x\equiv sp(\text{mod } m)}}^{p-1}x^k=\sum_{\substack{r\in\mathbb{Z}\\0\le sp-rm<p}}(sp-rm)^k=\sum_{\frac{(s-1)p}{m}<r\le\frac{sp}{m}}(sp-rm)^k \tag{2.6}$$
$$\equiv(-m)^k\sum_{\frac{(s-1)p}{m}<r\le\frac{sp}{m}}r^k \pmod p$$

and

$$\sum_{\substack{x=0\\x\equiv sp(\text{mod } m)}}^{p-1}(-1)^{\frac{x-sp}{m}}x^k=\sum_{\substack{r\in\mathbb{Z}\\0\le sp-rm<p}}(-1)^r(sp-rm)^k$$
$$=\sum_{\frac{(s-1)p}{m}<r\le\frac{sp}{m}}(-1)^r(sp-rm)^k \tag{2.7}$$
$$\equiv(-m)^k\sum_{\frac{(s-1)p}{m}<r\le\frac{sp}{m}}(-1)^r r^k \pmod p$$

Thus applying Corollary 2.1 we obtain the result.

Remark 2.2 In the case $s=1$, the first part of Corollary 2.2 is due to Lehmer ([L, p. 351]). In the case $k=p-2$, the first part of Corollary 2.2 can be deduced from [GS, p. 126].

Corollary 2.3. *Let p be a prime.*

(i) (Karpinski[K, UW]) *If* $p\equiv 3 \pmod 8$, *then* $\sum_{x=1}^{(p-3)/4}\left(\frac{x}{p}\right)=0.$

(ii) (Karpinski[K, UW]) *If* $p\equiv 5 \pmod 8$, *then* $\sum_{x=1}^{[p/6]}\left(\frac{x}{p}\right)=0.$

(iii) (Berndt[B, UW]) *If* $p\equiv 5 \pmod{24}$, *then* $\sum_{x=1}^{(p-5)/12}\left(\frac{x}{p}\right)=0.$

Proof. By Corollary 2.2 and the known fact $B_{2n+1}=0$, for $m\in\mathbb{N}$ with $p\nmid m$ we have

$$\begin{aligned}\sum_{x=1}^{[p/m]}\left(\frac{x}{p}\right)&\equiv\sum_{x=1}^{[p/m]}x^{\frac{p-1}{2}}\equiv\frac{(-1)^{\frac{p-1}{2}}}{\frac{p+1}{2}}\Big(B_{\frac{p+1}{2}}-B_{\frac{p+1}{2}}\Big(\Big\{\frac{p}{m}\Big\}\Big)\Big)\\&\equiv\begin{cases}-2B_{\frac{p+1}{2}}\left(\left\{\frac{p}{m}\right\}\right) \pmod p & \text{if } p\equiv 1 \pmod 4,\\ -2B_{\frac{p+1}{2}}+2B_{\frac{p+1}{2}}\left(\left\{\frac{p}{m}\right\}\right) \pmod p & \text{if } p\equiv 3 \pmod 4.\end{cases}\end{aligned}\tag{2.8}$$

It is well known that $B_{2n}(\frac34)=B_{2n}(\frac14)=(1-2^{2n-1})B_{2n}/2^{4n-1}$. Thus, if $p\equiv 3 \pmod 8$, by (2.8) we see that

$$\begin{aligned}\sum_{x=1}^{\frac{p-3}{4}}\left(\frac{x}{p}\right)&\equiv-2B_{\frac{p+1}{2}}+2B_{\frac{p+1}{2}}\left(\frac34\right)=\frac{1}{2^{p-1}}(1-2^{\frac{p-1}{2}})B_{\frac{p+1}{2}}-2B_{\frac{p+1}{2}}\\&\equiv\Big(1-\left(\frac{2}{p}\right)-2\Big)B_{\frac{p+1}{2}}=0 \pmod p.\end{aligned}$$

As $-\frac{p-3}{4}\le\sum_{x=1}^{\frac{p-3}{4}}\left(\frac{x}{p}\right)\le\frac{p-3}{4}$, we must have $\sum_{x=1}^{(p-3)/4}(\frac{x}{p})=0$. This proves (i).

Now we consider (ii). For $n\in\{0,1,2,\ldots\}$ and $m\in\mathbb{N}$ it is well known that (cf. [IR], [MOS])

$$B_n(1-x)=(-1)^nB_n(x)\quad\text{and}\quad\sum_{k=0}^{m-1}B_n\Big(x+\frac{k}{m}\Big)=m^{1-n}B_n(mx).\tag{2.9}$$

Thus

$$B_{\frac{p+1}{2}}\Big(\frac{1}{2n}\Big)+B_{\frac{p+1}{2}}\Big(\frac{1}{2n}+\frac12\Big)=2^{-\frac{p-1}{2}}B_{\frac{p+1}{2}}\Big(\frac1n\Big)$$

and so

$$B_{\frac{p+1}{2}}\Big(\frac{1}{2n}\Big)\equiv\left(\frac{2}{p}\right)B_{\frac{p+1}{2}}\Big(\frac1n\Big)-(-1)^{\frac{p+1}{2}}B_{\frac{p+1}{2}}\Big(\frac{n-1}{2n}\Big) \pmod p.\tag{2.10}$$

Since $p\equiv 5 \pmod 8$, taking $n=3$ in (2.10) we find

$$B_{\frac{p+1}{2}}\Big(\frac16\Big)\equiv-B_{\frac{p+1}{2}}\Big(\frac13\Big)+B_{\frac{p+1}{2}}\Big(\frac13\Big)=0 \pmod p.\tag{2.11}$$

This together with (2.8) and (2.9) yields

$$\sum_{x=1}^{[p/6]}\Big(\frac{x}{p}\Big)\equiv -2B_{\frac{p+1}{2}}\Big(\Big\{\frac{p}{6}\Big\}\Big)=-2\Big(\frac{p}{3}\Big)B_{\frac{p+1}{2}}\Big(\frac{1}{6}\Big)\equiv 0 \pmod p.$$

As $|\sum_{x=1}^{[p/6]}(\frac{x}{p})|\le [\frac{p}{6}]$ we have $\sum_{x=1}^{[p/6]}(\frac{x}{p})=0$. This proves (ii).

Finally we consider (iii). Assume $p\equiv 5 \pmod{24}$. By (2.10) and (2.11) we have

$$B_{\frac{p+1}{2}}\Big(\frac{1}{12}\Big)\equiv\Big(\frac{2}{p}\Big)B_{\frac{p+1}{2}}\Big(\frac{1}{6}\Big)+B_{\frac{p+1}{2}}\Big(\frac{5}{12}\Big)\equiv B_{\frac{p+1}{2}}\Big(\frac{5}{12}\Big) \pmod p.$$

On the other hand, by (2.9) we have

$$\begin{aligned}B_{\frac{p+1}{2}}\Big(\frac{1}{12}\Big)+B_{\frac{p+1}{2}}\Big(\frac{5}{12}\Big)&=3^{-\frac{p-1}{2}}B_{\frac{p+1}{2}}\Big(\frac{1}{4}\Big)-B_{\frac{p+1}{2}}\Big(\frac{9}{12}\Big)\\&\equiv\Big(\frac{3}{p}\Big)B_{\frac{p+1}{2}}\Big(\frac{1}{4}\Big)-(-1)^{\frac{p+1}{2}}B_{\frac{p+1}{2}}\Big(\frac{1}{4}\Big)\\&=0 \pmod p.\end{aligned}$$

Thus $B_{\frac{p+1}{2}}(\frac{1}{12})\equiv B_{\frac{p+1}{2}}(\frac{5}{12})\equiv 0 \pmod p$. Now applying (2.8) we see that

$$\sum_{x=1}^{[p/12]}\Big(\frac{x}{p}\Big)\equiv -2B_{\frac{p+1}{2}}\Big(\Big\{\frac{p}{12}\Big\}\Big)=-2B_{\frac{p+1}{2}}\Big(\frac{5}{12}\Big)\equiv 0 \pmod p.$$

This yields (iii) and so the corollary is proved.

Corollary 2.4. *Suppose $p,q,m\in\mathbb{N}$, $n\in\mathbb{Z}$, $\gcd(p,m)=1$ and $q\le m$. For $r\in\mathbb{Z}$ let $A_r(m,p)$ be the least positive solution of the congruence $px\equiv r \pmod m$. Then*

$$\big|\{r:\ A_r(m,p)\le q,\ r\in\mathbb{Z},\ -n\le r\le p-1-n\}\big|=\Big[\frac{pq+n}{m}\Big]-\Big[\frac{n}{m}\Big].$$

Proof. Using Theorem 2.1 we see that

$$\begin{aligned}&\big|\{r:\ A_r(m,p)\le q,\ r\in\mathbb{Z},\ -n\le r\le p-1-n\}\big|\\&=\sum_{x=1}^{q}\sum_{\substack{r=-n\\ r\equiv px \pmod m}}^{p-1-n}1=\sum_{x=1}^{q}\sum_{\substack{s=0\\ s\equiv px+n \pmod m}}^{p-1}1\\&=\sum_{x=1}^{q}\Big(B_1\Big(\frac{p}{m}+\Big\{\frac{px+n-p}{m}\Big\}\Big)-B_1\Big(\Big\{\frac{px+n}{m}\Big\}\Big)\Big)\\&=\sum_{x=1}^{q}\Big(\frac{p}{m}+\Big\{\frac{p(x-1)+n}{m}\Big\}-\Big\{\frac{px+n}{m}\Big\}\Big)\\&=\frac{pq}{m}+\Big\{\frac{n}{m}\Big\}-\Big\{\frac{pq+n}{m}\Big\}=\frac{pq+n}{m}-\Big\{\frac{pq+n}{m}\Big\}-\Big(\frac{n}{m}-\Big\{\frac{n}{m}\Big\}\Big)\\&=\Big[\frac{pq+n}{m}\Big]-\Big[\frac{n}{m}\Big].\end{aligned}$$

This proves the corollary.

Theorem 2.2. *Let $m, s \in \mathbb{N}$ and let p be an odd prime not dividing m. Then*

$$(-1)^s \frac{m}{p} \sum_{\substack{k=1 \\ k \equiv sp \pmod m}}^{p-1} \binom{p}{k}$$

$$\equiv \sum_{\frac{(s-1)p}{m} < k < \frac{sp}{m}} \frac{(-1)^{km}}{k}$$

$$\equiv \begin{cases} B_{p-1}\big(\big\{\frac{(s-1)p}{m}\big\}\big) - B_{p-1}\big(\big\{\frac{sp}{m}\big\}\big) \pmod p & \text{if } 2 \mid m, \\ \frac{1}{2}\big((-1)^{[\frac{(s-1)p}{m}]} E_{p-2}\big(\big\{\frac{(s-1)p}{m}\big\}\big) - (-1)^{[\frac{sp}{m}]} E_{p-2}\big(\big\{\frac{sp}{m}\big\}\big)\big) \pmod p & \text{if } 2 \nmid m. \end{cases}$$

Proof. Let $r \in \mathbb{Z}$. Since $\binom{p-1}{j} \equiv (-1)^j \pmod p$ for $j \in \{0, 1, \ldots, p-1\}$ we see that

$$\frac{1}{p} \sum_{\substack{k=1 \\ k \equiv r \pmod m}}^{p-1} \binom{p}{k} = \sum_{\substack{k=1 \\ k \equiv r \pmod m}}^{p-1} \frac{1}{k}\binom{p-1}{k-1} \equiv \sum_{\substack{k=1 \\ k \equiv r \pmod m}}^{p-1} \frac{(-1)^{k-1}}{k}$$

$$= \begin{cases} (-1)^{r-1} \sum\limits_{\substack{k=1 \\ k \equiv r \pmod m}}^{p-1} \frac{1}{k} \pmod p & \text{if } 2 \mid m, \\ (-1)^{r-1} \sum\limits_{\substack{k=1 \\ k \equiv r \pmod m}}^{p-1} (-1)^{\frac{k-r}{m}} \frac{1}{k} \pmod p & \text{if } 2 \nmid m. \end{cases}$$

Putting this together with (2.4) and (2.5) we see that

$$\frac{1}{p} \sum_{\substack{k=1 \\ k \equiv r \pmod m}}^{p-1} \binom{p}{k}$$

$$\equiv \begin{cases} \frac{(-1)^r}{m}\big(B_{p-1}\big(\big\{\frac{r-p}{m}\big\}\big) - B_{p-1}\big(\big\{\frac{r}{m}\big\}\big)\big) \pmod p & \text{if } 2 \mid m, \\ \frac{(-1)^r}{2m}\big((-1)^{[\frac{r-p}{m}]} E_{p-2}\big(\big\{\frac{r-p}{m}\big\}\big) - (-1)^{[\frac{r}{m}]} E_{p-2}\big(\big\{\frac{r}{m}\big\}\big)\big) \pmod p & \text{if } 2 \nmid m. \end{cases}$$

Taking $r = sp$ we obtain

$$(-1)^s \frac{m}{p} \sum_{\substack{k=1 \\ k \equiv sp \pmod m}}^{p-1} \binom{p}{k}$$

$$\equiv \begin{cases} B_{p-1}\big(\big\{\frac{(s-1)p}{m}\big\}\big) - B_{p-1}\big(\big\{\frac{sp}{m}\big\}\big) \pmod p & \text{if } 2 \mid m, \\ \frac{1}{2}\big((-1)^{[\frac{(s-1)p}{m}]} E_{p-2}\big(\big\{\frac{(s-1)p}{m}\big\}\big) - (-1)^{[\frac{sp}{m}]} E_{p-2}\big(\big\{\frac{sp}{m}\big\}\big)\big) \pmod p & \text{if } 2 \nmid m. \end{cases}$$

On the other hand, putting $k = p-2$ in Corollary 2.2 we see that

$$B_{p-1}\Big(\Big\{\frac{(s-1)p}{m}\Big\}\Big) - B_{p-1}\Big(\Big\{\frac{sp}{m}\Big\}\Big) \equiv \sum_{\frac{(s-1)p}{m} < r < \frac{sp}{m}} \frac{1}{r} \pmod p$$

and

$$(-1)^{[\frac{(s-1)p}{m}]}E_{p-2}\Big(\Big\{\frac{(s-1)p}{m}\Big\}\Big)-(-1)^{[\frac{sp}{m}]}E_{p-2}\Big(\Big\{\frac{sp}{m}\Big\}\Big)$$
$$\equiv 2\sum_{\frac{(s-1)p}{m}<r<\frac{sp}{m}}\frac{(-1)^r}{r} \pmod p.$$

Now combining the above we prove the theorem.

Corollary 2.5. *Let $m,n\in\mathbb{N}$ and let p be an odd prime not dividing m.*

(i) *If $2\mid m$, then*

$$B_{p-1}\Big(\Big\{\frac{np}{m}\Big\}\Big)-B_{p-1}\equiv\frac{m}{p}\sum_{s=1}^{n}(-1)^{s-1}\sum_{\substack{k=1\\ k\equiv sp\pmod m}}^{p-1}\binom{p}{k} \pmod p.$$

(ii) *If $2\nmid m$, then*

$$(-1)^{[\frac{np}{m}]}E_{p-2}\Big(\Big\{\frac{np}{m}\Big\}\Big)+\frac{2^p-2}{p}\equiv\frac{2m}{p}\sum_{s=1}^{n}(-1)^{s-1}\sum_{\substack{k=1\\ k\equiv sp\pmod m}}^{p-1}\binom{p}{k} \pmod p.$$

Proof. It is well known that $pB_{p-1}\equiv p-1 \pmod p$. Thus, by (1.2) we have $E_{p-2}(0)=2(1-2^{p-1})B_{p-1}/(p-1)\equiv-(2^p-2)/p \pmod p$. Note that $\sum_{s=1}^{n}(f(s)-f(s-1))=f(n)-f(0)$. Then the result follows from Theorem 2.2 and the above immediately.

Combining Theorem 2.2, Corollary 2.5 with the formulae for $\sum_{k\equiv r\pmod m}\binom{p}{k}$ in the cases $m=3,4,5,6,8,9,10,12$ (see [S1,S2,S3,SS,Su1]) we may deduce many useful results, which had been given in [GS] and [S3].

3. Some congruences for $h(-3p),h(-5p),h(-8p),h(-12p) \pmod p$.

Let $\{S_n\}$ be defined by

$$S_0=1 \quad\text{and}\quad S_n=1-\sum_{k=0}^{n-1}\binom{n}{k}2^{2n-2k-1}S_k \quad (n\ge 1). \tag{3.1}$$

Then clearly $S_n\in\mathbb{Z}$. The first few S_n are shown below:

$$S_1=-1,\ S_2=-3,\ S_3=11,\ S_4=57,\ S_5=-361,\ S_6=-2763.$$

Theorem 3.1. *Let p be an odd prime. Then*

$$h(-8p)\equiv E_{\frac{p-1}{2}}\Big(\frac14\Big)\equiv S_{\frac{p-1}{2}} \pmod p.$$

Proof. From [UW, p. 58] we know that

$$h(-8p) = 2 \sum_{\substack{a=1 \\ a\equiv 1 \pmod 4}}^{p-1} \Big(\frac{8p}{a}\Big). \tag{3.2}$$

Thus applying Corollary 2.1 in the case $r = 1$, $m = 4$ and $k = \frac{p-1}{2}$ we see that

$$\begin{aligned} h(-8p) &= 2 \sum_{\substack{a=0 \\ a\equiv 1 \pmod 4}}^{p-1} \Big(\frac{2}{a}\Big)\Big(\frac{a}{p}\Big) \equiv 2 \sum_{\substack{a=0 \\ a\equiv 1 \pmod 4}}^{p-1} (-1)^{\frac{a-1}{4}} a^{\frac{p-1}{2}} \\ &\equiv -4^{\frac{p-1}{2}} \Big((-1)^{[\frac{1-p}{4}]} E_{\frac{p-1}{2}}\Big(\Big\{\frac{1-p}{4}\Big\}\Big) - E_{\frac{p-1}{2}}\Big(\frac{1}{4}\Big)\Big) \pmod p. \end{aligned}$$

Since $E_{2n}(0) = \frac{2}{2n+1}(B_{2n+1} - 2^{2n+1}B_{2n+1}) = 0$ by (1.2), we see that

$$E_{\frac{p-1}{2}}\Big(\Big\{\frac{1-p}{4}\Big\}\Big) = \begin{cases} E_{2n}(0) = 0 & \text{if } p = 4n+1, \\ E_{2n-1}(\frac{1}{2}) = 2^{1-2n}E_{2n-1} = 0 & \text{if } p = 4n-1. \end{cases}$$

Thus

$$h(-8p) \equiv 4^{\frac{p-1}{2}} E_{\frac{p-1}{2}}\Big(\frac{1}{4}\Big) \equiv E_{\frac{p-1}{2}}\Big(\frac{1}{4}\Big) \pmod p.$$

Let $S'_n = 4^n E_n(\frac{1}{4})$. Now we show that $S_n = S'_n$ for $n \ge 0$. By (1.1) we have

$$4^{-n}S'_n + \sum_{k=0}^{n} \binom{n}{k} 4^{-k} S'_k = 2 \cdot 4^{-n} \quad \text{and so} \quad S'_n + \sum_{k=0}^{n} \binom{n}{k} 4^{n-k} S'_k = 2.$$

That is, $S'_n = 1 - \sum_{k=0}^{n-1} \binom{n}{k} 2^{2n-2k-1} S'_k$. Since $S'_0 = S_0 = 1$ we see that $S'_n = S_n$. That is,

$$S_n = 4^n E_n\Big(\frac{1}{4}\Big). \tag{3.3}$$

Hence $S_{\frac{p-1}{2}} = 4^{\frac{p-1}{2}} E_{\frac{p-1}{2}}(\frac{1}{4}) \equiv h(-8p) \pmod p$. This proves the theorem.

Corollary 3.1. *Let p be an odd prime. Then $p \nmid S_{\frac{p-1}{2}}$.*

Proof. From (3.2) we have $1 < h(-8p) < p$. Thus the result follows from Theorem 3.1.

Remark 3.1 Since $S_n = 4^n E_n(\frac{1}{4})$, by (1.2) and the binomial inversion formula we have

$$S_n = \sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} 2^r E_r \quad \text{and} \quad \sum_{r=0}^{n} \binom{n}{r} S_r = 2^n E_n. \tag{3.4}$$

Theorem 3.2. *Let p be a prime greater than* 3.

(i) *If $p \equiv 1 \pmod 4$, then*

$$h(-3p) \equiv \begin{cases} -4B_{\frac{p+1}{2}}\left(\frac{1}{3}\right) \pmod p & \text{if } p \equiv 1 \pmod{12}, \\ 4B_{\frac{p+1}{2}}\left(\frac{1}{3}\right) \pmod p & \text{if } p \equiv 5 \pmod{12}. \end{cases}$$

(ii) *If $p \equiv 3 \pmod 4$, then*

$$h(-12p) \equiv \begin{cases} 8B_{\frac{p+1}{2}}\left(\frac{1}{12}\right) \pmod p & \text{if } p \equiv 7 \pmod{24}, \\ -8B_{\frac{p+1}{2}}\left(\frac{1}{12}\right) \pmod p & \text{if } p \equiv 11 \pmod{12}, \\ 8B_{\frac{p+1}{2}}\left(\frac{1}{12}\right) + 8B_{\frac{p+1}{2}} \pmod p & \text{if } p \equiv 19 \pmod{24} \end{cases}$$

and

$$h(-5p) \equiv \begin{cases} -8B_{\frac{p+1}{2}}\left(\frac{1}{5}\right) \pmod p & \text{if } p \equiv 11, 19 \pmod{20}, \\ 8B_{\frac{p+1}{2}}\left(\frac{1}{5}\right) + 4B_{\frac{p+1}{2}} \pmod p & \text{if } p \equiv 3, 7 \pmod{20}. \end{cases}$$

.

Proof. We first assume $p \equiv 1 \pmod 4$. From [UW, p. 40] or [B] we have

$$h(-3p) = 2\sum_{x=1}^{[p/3]} \left(\frac{p}{x}\right).$$

Thus applying (2.8), (2.9) and the quadratic reciprocity law we see that

$$h(-3p) = 2\sum_{x=1}^{[p/3]} \left(\frac{x}{p}\right) \equiv -4B_{\frac{p+1}{2}}\left(\left\{\frac{p}{3}\right\}\right) = -4\left(\frac{p}{3}\right)B_{\frac{p+1}{2}}\left(\frac{1}{3}\right) \pmod p.$$

This proves (i).

Now let us consider (ii). Assume $p \equiv 3 \pmod 4$. From [UW, pp. 3-5] we have

$$h(-12p) = \begin{cases} 4\sum_{\frac{p}{12}<x<\frac{2p}{12}} \left(\frac{x}{p}\right) & \text{if } p \equiv 7, 11, 23 \pmod{24}, \\ 4\sum_{\frac{4p}{12}<x<\frac{5p}{12}} \left(\frac{x}{p}\right) & \text{if } p \equiv 19 \pmod{24}. \end{cases}$$

By Corollary 2.2 and the fact $B_{2n}(x) = B_{2n}(1-x)$ we find

$$\sum_{\frac{p}{12}<x<\frac{2p}{12}} \left(\frac{x}{p}\right) \equiv \sum_{\frac{p}{12}<x\le\frac{2p}{12}} x^{\frac{p-1}{2}} \equiv -2\Big(B_{\frac{p+1}{2}}\left(\left\{\frac{p}{12}\right\}\right) - B_{\frac{p+1}{2}}\left(\frac{1}{6}\right)\Big) \pmod p$$

and

$$\sum_{\frac{4p}{12}<x<\frac{5p}{12}} \left(\frac{x}{p}\right) \equiv \sum_{\frac{4p}{12}<x\le\frac{5p}{12}} x^{\frac{p-1}{2}} \equiv -2\Big(B_{\frac{p+1}{2}}\left(\frac{1}{3}\right) - B_{\frac{p+1}{2}}\left(\left\{\frac{5p}{12}\right\}\right)\Big) \pmod p.$$

Thus

$$h(-12p) \equiv \begin{cases} -8\big(B_{\frac{p+1}{2}}\big(\frac{5}{12}\big) - B_{\frac{p+1}{2}}\big(\frac{1}{6}\big)\big) \pmod p & \text{if } p \equiv 7 \pmod{24}, \\ -8\big(B_{\frac{p+1}{2}}\big(\frac{1}{12}\big) - B_{\frac{p+1}{2}}\big(\frac{1}{6}\big)\big) \pmod p & \text{if } p \equiv 11 \pmod{12}, \\ -8\big(B_{\frac{p+1}{2}}\big(\frac{1}{3}\big) - B_{\frac{p+1}{2}}\big(\frac{1}{12}\big)\big) \pmod p & \text{if } p \equiv 19 \pmod{24}. \end{cases}$$

By (2.10) we have

$$B_{\frac{p+1}{2}}\Big(\frac{1}{12}\Big) \equiv \Big(\frac{2}{p}\Big)B_{\frac{p+1}{2}}\Big(\frac{1}{6}\Big) - B_{\frac{p+1}{2}}\Big(\frac{5}{12}\Big) \pmod p.$$

Thus, if $p \equiv 7 \pmod{24}$, then $h(-12p) \equiv 8(B_{\frac{p+1}{2}}\big(\frac{1}{6}\big) - B_{\frac{p+1}{2}}\big(\frac{5}{12}\big)) \equiv 8B_{\frac{p+1}{2}}\big(\frac{1}{12}\big) \pmod p$. It is well known that ([GS])

$$B_{2n}\Big(\frac{1}{3}\Big) = \frac{3^{1-2n}-1}{2}B_{2n} \quad \text{and} \quad B_{2n}\Big(\frac{1}{6}\Big) = \frac{(2^{1-2n}-1)(3^{1-2n}-1)}{2}B_{2n}.$$

Thus

$$B_{\frac{p+1}{2}}\Big(\frac{1}{3}\Big) = \frac{1}{2}\Big(3^{-\frac{p-1}{2}}-1\Big)B_{\frac{p+1}{2}} \equiv \frac{1}{2}\Big(\Big(\frac{3}{p}\Big)-1\Big)B_{\frac{p+1}{2}} \pmod p$$

and

$$B_{\frac{p+1}{2}}\Big(\frac{1}{6}\Big) = \frac{(2^{-\frac{p-1}{2}}-1)(3^{-\frac{p-1}{2}}-1)}{2}B_{\frac{p+1}{2}} \equiv \frac{1}{2}\Big(\Big(\frac{2}{p}\Big)-1\Big)\Big(\Big(\frac{3}{p}\Big)-1\Big)B_{\frac{p+1}{2}} \pmod p.$$

If $p \equiv 11 \pmod{12}$, then $\big(\frac{3}{p}\big) = 1$ and so $B_{\frac{p+1}{2}}\big(\frac{1}{6}\big) \equiv 0 \pmod p$. Hence $h(-12p) \equiv -8B_{\frac{p+1}{2}}\big(\frac{1}{12}\big) \pmod p$. If $p \equiv 19 \pmod{24}$, then $\big(\frac{3}{p}\big) = -1$ and so $B_{\frac{p+1}{2}}\big(\frac{1}{3}\big) \equiv -B_{\frac{p+1}{2}} \pmod p$. Thus $h(-12p) \equiv 8(B_{\frac{p+1}{2}}\big(\frac{1}{12}\big) + B_{\frac{p+1}{2}}) \pmod p$.

Finally we consider $h(-5p) \pmod p$. From [UW, p. 40] or [B] we have

$$h(-5p) = 2\sum_{\frac{p}{5}<a<\frac{2p}{5}}\Big(\frac{-p}{a}\Big).$$

Observe that $\big(\frac{-p}{a}\big) = \big(\frac{a}{p}\big)$ by the quadratic reciprocity law. Thus applying Corollary 2.2 and (2.9) we obtain

$$\begin{aligned} h(-5p) &= 2\sum_{\frac{p}{5}<a<\frac{2p}{5}}\Big(\frac{a}{p}\Big) \equiv 2\sum_{\frac{p}{5}<a<\frac{2p}{5}} a^{\frac{p-1}{2}} \\ &\equiv 2\cdot\frac{(-1)^{\frac{p-1}{2}}}{(p+1)/2}\Big(B_{\frac{p+1}{2}}\Big(\Big\{\frac{p}{5}\Big\}\Big) - B_{\frac{p+1}{2}}\Big(\Big\{\frac{2p}{5}\Big\}\Big)\Big) \\ &\equiv -4\Big(\frac{p}{5}\Big)\Big(B_{\frac{p+1}{2}}\Big(\frac{1}{5}\Big) - B_{\frac{p+1}{2}}\Big(\frac{2}{5}\Big)\Big) \pmod p. \end{aligned}$$

From (2.9) we see that

$$B_{\frac{p+1}{2}}+2B_{\frac{p+1}{2}}\Big(\frac15\Big)+2B_{\frac{p+1}{2}}\Big(\frac25\Big)=\sum_{k=0}^{4}B_{\frac{p+1}{2}}\Big(\frac k5\Big)=5^{-\frac{p-1}{2}}B_{\frac{p+1}{2}}$$

and so

$$B_{\frac{p+1}{2}}\Big(\frac15\Big)+B_{\frac{p+1}{2}}\Big(\frac25\Big)\equiv\frac12\Big(\Big(\frac p5\Big)-1\Big)B_{\frac{p+1}{2}} \pmod p.$$

Thus

$$\begin{aligned}h(-5p)&\equiv-4\Big(\frac p5\Big)\Big(2B_{\frac{p+1}{2}}\Big(\frac15\Big)+\frac12\Big(1-\Big(\frac p5\Big)\Big)B_{\frac{p+1}{2}}\Big)\\&=\begin{cases}-8B_{\frac{p+1}{2}}\big(\frac15\big)\pmod p & \text{if } p\equiv 11,19\pmod{20},\\ 8B_{\frac{p+1}{2}}\big(\frac15\big)+4B_{\frac{p+1}{2}}\pmod p & \text{if } p\equiv 3,7\pmod{20}.\end{cases}\end{aligned}$$

The proof is now complete.

When d is a negative discriminant, it is known that $1\le h(d)<p$. Thus, from Theorem 3.2 we deduce the following result.

Corollary 3.2. *Let p be a prime.*

(i) *If $p\equiv 1\pmod 4$, then $B_{\frac{p+1}{2}}(\frac13)\not\equiv 0\pmod p$.*

(ii) *If $p\equiv 7,11,23\pmod{24}$, then $B_{\frac{p+1}{2}}(\frac1{12})\not\equiv 0\pmod p$.*

(iii) *If $p\equiv 11,19\pmod{20}$, then $B_{\frac{p+1}{2}}(\frac15)\not\equiv 0\pmod p$.*

Remark 3.2 For $n=0,1,\ldots$ it is well known that $\sum_{k=0}^{n}\binom nk\frac{1}{n-k+1}B_k(x)=x^n$. From this we deduce that if $m\in\mathbb N$ and $a_n=m^nB_n(\frac1m)$, then $\sum_{k=0}^{n}\binom{n+1}{k}m^{n-k}a_k=n+1$.

4. p-regular functions.

For a prime p, in [S5] the author introduced the notion of p-regular functions. If $f(k)$ is a complex number congruent to an algebraic integer modulo p for any given nonnegative integer k and $\sum_{k=0}^{n}\binom nk(-1)^kf(k)\equiv 0\pmod{p^n}$ for all $n\in\mathbb N$, then f is called a p-regular function. If f and g are p-regular functions, in [S5] the author showed that $f\cdot g$ is also a p-regular function. Thus we see that p-regular functions form a ring. In the section we discuss further properties of p-regular functions.

Suppose $n\in\mathbb N$ and $k\in\{0,1,\ldots,n\}$. Let $s(n,k)$ be the unsigned Stirling number of the first kind and $S(n,k)$ be the Stirling number of the second kind defined by

$$x(x-1)\cdots(x-n+1)=\sum_{k=0}^{n}(-1)^{n-k}s(n,k)x^k$$

and

$$x^n=\sum_{k=0}^{n}S(n,k)x(x-1)\cdots(x-k+1).$$

For our convenience we also define $s(n,k) = S(n,k) = 0$ for $k > n$. For $m \in \mathbb{N}$ it is well known that

$$\sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} r^m = n! S(m,n) \tag{4.1}$$

In particular, taking $m = n$ we have the following Euler's identity

$$\sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} r^n = n!\,. \tag{4.2}$$

Lemma 4.1. *Let x, d be variables, $m, n \in \mathbb{N}$ and $i \in \mathbb{Z}$ with $i \geq 0$. Then*

$$\begin{aligned}
&\sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} \binom{rx+d}{m} r^i \\
&= \frac{n!}{m!} \sum_{j=n-i}^{m} \Big(\sum_{k=j}^{m} \binom{k}{j} (-1)^{m-k} s(m,k) d^{k-j} \Big) S(i+j,n) x^j.
\end{aligned}$$

In particular we have

$$\sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} \binom{rx}{m} r^i = \frac{n!}{m!} \sum_{j=n-i}^{m} (-1)^{m-j} s(m,j) S(i+j,n) x^j.$$

Proof. Since

$$\begin{aligned}
m! \binom{rx+d}{m} &= (rx+d)(rx+d-1)\cdots(rx+d-m+1) \\
&= \sum_{k=0}^{m} (-1)^{m-k} s(m,k) (rx+d)^k \\
&= \sum_{k=0}^{m} (-1)^{m-k} s(m,k) \sum_{j=0}^{k} \binom{k}{j} (rx)^j d^{k-j} \\
&= \sum_{j=0}^{m} \Big(\sum_{k=j}^{m} \binom{k}{j} (-1)^{m-k} s(m,k) d^{k-j} \Big) r^j x^j,
\end{aligned}$$

we have

$$\begin{aligned}
&\sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} \binom{rx+d}{m} r^i \\
&= \frac{1}{m!} \sum_{j=0}^{m} \Big(\sum_{k=j}^{m} \binom{k}{j} (-1)^{m-k} s(m,k) d^{k-j} \Big) x^j \cdot \sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} r^{i+j}.
\end{aligned}$$

Now applying (4.1) we obtain the result.

Lemma 4.2. *Let p be a prime and $m,n\in\mathbb{N}$. Then*

$$\frac{m!s(n,m)}{n!}p^{n-m}\in\mathbb{Z}_p \quad \text{and} \quad \frac{m!S(n,m)}{n!}p^{n-m}\in\mathbb{Z}_p.$$

Moreover, if $m<n$, we have

$$\frac{m!s(n,m)}{n!}p^{n-m}\equiv\frac{m!S(n,m)}{n!}p^{n-m}\equiv 0 \pmod p \quad \text{for } p>2$$

and

$$\frac{m!s(n,m)}{n!}2^{n-m}\equiv\binom{m}{n-m} \pmod 2.$$

Proof. It is well known that

$$\frac{(\mathrm{e}^x-1)^m}{m!}=\sum_{n=m}^{\infty}S(n,m)\frac{x^n}{n!}.$$

Thus, applying the multinomial theorem we see that

$$(\mathrm{e}^x-1)^m=\Big(\sum_{k=1}^{\infty}\frac{x^k}{k!}\Big)^m=\sum_{n=m}^{\infty}\Bigg(\sum_{\substack{k_1+k_2+\cdots+k_n=m\\ k_1+2k_2+\cdots+nk_n=n}}\frac{m!}{k_1!k_2!\cdots k_n!}\prod_{r=1}^{n}\frac{1}{r!^{k_r}}\Bigg)x^n$$

and so

$$S(n,m)=\sum_{\substack{k_1+k_2+\cdots+k_n=m\\ k_1+2k_2+\cdots+nk_n=n}}\frac{n!}{1!^{k_1}k_1!2!^{k_2}k_2!\cdots n!^{k_n}k_n!}. \tag{4.3}$$

Hence

$$\frac{m!S(n,m)}{n!}p^{n-m}=\sum_{\substack{k_1+k_2+\cdots+k_n=m\\ k_1+2k_2+\cdots+nk_n=n}}\frac{(k_1+k_2+\cdots+k_n)!}{k_1!k_2!\cdots k_n!}\prod_{r=1}^{n}\Big(\frac{p^{r-1}}{r!}\Big)^{k_r}.$$

From [S5, pp. 196-197] we also have

$$s(n,m)=\sum_{\substack{k_1+k_2+\cdots+k_n=m\\ k_1+2k_2+\cdots+nk_n=n}}\frac{n!}{1^{k_1}k_1!2^{k_2}k_2!\cdots n^{k_n}k_n!} \tag{4.4}$$

and

$$\frac{m!s(n,m)}{n!}p^{n-m}=\sum_{\substack{k_1+k_2+\cdots+k_n=m\\ k_1+2k_2+\cdots+nk_n=n}}\frac{(k_1+k_2+\cdots+k_n)!}{k_1!k_2!\cdots k_n!}\prod_{r=1}^{n}\Big(\frac{p^{r-1}}{r}\Big)^{k_r}.$$

It is known that $(k_1+\cdots+k_n)!/(k_1!\cdots k_n!)\in\mathbb{Z}$. For $r\in\mathbb{N}$ we know that if $p^\alpha \parallel r!$(that is $p^\alpha\mid r!$ but $p^{\alpha+1}\nmid r!$), then $\alpha=\sum_{i=1}^\infty\left[\frac{r}{p^i}\right]\le\left[\frac{r}{p}\right]$. Thus $p^{r-1}/r,\ p^{r-1}/r!\in\mathbb{Z}_p$. For $p>2$ we see that $p^{r-1}/r\equiv p^{r-1}/r!\equiv 0 \pmod p$ for $r>1$. Hence the result follows from the above. For $p=2$ we see that $2^{r-1}/r\equiv 0\pmod 2$ for $r>2$. Thus

$$\frac{m!s(n,m)}{n!}2^{n-m}\equiv\sum_{\substack{k_1+k_2=m\\k_1+2k_2=n}}\frac{(k_1+k_2)!}{k_1!k_2!}=\binom{m}{n-m}\pmod 2.$$

Summarizing the above we prove the lemma.

From Lemma 4.1 we have the following identities, which are generalizations of Euler's identity.

Theorem 4.1. *Let x,d be variables and $m,n\in\mathbb{N}$.*

(i) *If $m\le n$, then*

$$\sum_{r=0}^n\binom{n}{r}(-1)^{n-r}\binom{rx+d}{m}r^{n-m}=\frac{n!}{m!}x^m.$$

In particular, when $m=n$ we have

$$\sum_{r=0}^n\binom{n}{r}(-1)^{n-r}\binom{rx+d}{n}=x^n.$$

(ii) *If $m\le n+1$, then*

$$\sum_{r=0}^n\binom{n}{r}(-1)^{n-r}\binom{rx+d}{m}r^{n+1-m}=\frac{n!}{m!}\Big(\frac{n(n+1)}{2}x^m-\frac{m(m-1-2d)}{2}x^{m-1}\Big).$$

In particular, when $m=n+1$ we have

$$\sum_{r=0}^n\binom{n}{r}(-1)^{n-r}\binom{rx+d}{n+1}=\Big(d+\frac{n(x-1)}{2}\Big)x^n.$$

Proof. Observe that $s(m,m)=1$ and $S(n,n)=1$. Putting $i=n-m$ in Lemma 4.1 we obtain (i). By (4.3) and (4.4) we have

$$s(n,n-1)=S(n,n-1)=n(n-1)/2\quad\text{for}\quad n=2,3,4,\ldots$$

Thus applying Lemma 4.1 we see that if $m\le n+1$, then

$$\begin{aligned}&\sum_{r=0}^n\binom{n}{r}(-1)^{n-r}\binom{rx+d}{m}r^{n+1-m}\\&=\frac{n!}{m!}\sum_{j=m-1}^m\Big(\sum_{k=j}^m\binom{k}{j}(-1)^{m-k}s(m,k)d^{k-j}\Big)S(n+1-m+j,n)x^j\\&=\frac{n!}{m!}\left(S(n+1,n)x^m+\sum_{k=m-1}^m\binom{k}{m-1}(-1)^{m-k}s(m,k)d^{k-(m-1)}x^{m-1}\right)\\&=\frac{n!}{m!}\Big(\frac{n(n+1)}{2}x^m+\Big(dm-\frac{m(m-1)}{2}\Big)x^{m-1}\Big).\end{aligned}$$

This yields (ii) and so the theorem is proved.

Corollary 4.1. *Let p be an odd prime, $m \in \mathbb{Z}$ and $d \in \{0,1,\dots,p-1\}$. Then $m^p \equiv m \pmod p$ and*

$$\frac{m^p-m}{p} \equiv \sum_{k=1}^{p-1}\frac{1}{k}\Big[\frac{km+d}{p}\Big] + m\sum_{k=1}^{d}\frac{1}{k} \pmod p.$$

Proof. From Theorem 4.1(i) we have

$$\begin{aligned} m^p &= \sum_{k=0}^{p}\binom{p}{k}(-1)^{p-k}\binom{km+d}{p} \\ &= \binom{mp+d}{p} + \sum_{k=1}^{p-1}\binom{p}{k}(-1)^{p-k}\binom{km+d}{p}. \end{aligned}$$

As $\sum_{k=1}^{p-1}\frac{1}{k} \equiv 0 \pmod p$, we see that

$$\begin{aligned} \binom{mp+d}{p} &= \frac{(mp+d)(mp+d-1)\cdots(mp+d-p+1)}{p!} \\ &= \frac{mp}{p}\cdot\frac{(mp+1)\cdots(mp+d)((m-1)p+d+1)\cdots((m-1)p+p-1)}{(p-1)!} \\ &\equiv m\Big(1+mp\sum_{k=1}^{d}\frac{1}{k}+(m-1)p\sum_{k=d+1}^{p-1}\frac{1}{k}\Big) \\ &\equiv m\Big(1+mp\sum_{k=1}^{d}\frac{1}{k}-(m-1)p\sum_{k=1}^{d}\frac{1}{k}\Big) \\ &= m\Big(1+p\sum_{k=1}^{d}\frac{1}{k}\Big) \pmod{p^2}. \end{aligned}$$

Let r_k be the least nonnegative residue of $km+d$ modulo p. For $k \in \{1,2,\dots,p-1\}$ we see that

$$\binom{p}{k} = \frac{p(p-1)\cdots(p-k+1)}{k!} \equiv \frac{(-1)^{k-1}}{k}p \pmod{p^2}.$$

Thus,

$$\begin{aligned} &\sum_{k=1}^{p-1}\binom{p}{k}(-1)^{p-k}\binom{km+d}{p} \\ &\equiv \sum_{k=1}^{p-1}\frac{p}{k}\cdot\frac{(km+d)(km+d-1)\cdots(km+d-p+1)}{p!} \\ &= p\sum_{k=1}^{p-1}\frac{1}{k}\cdot\frac{km+d-r_k}{p}\cdot\frac{1}{(p-1)!}\prod_{\substack{i=0\\ i\neq r_k}}^{p-1}(km+d-i) \\ &\equiv p\sum_{k=1}^{p-1}\frac{1}{k}\cdot\frac{km+d-r_k}{p} = p\sum_{k=1}^{p-1}\frac{1}{k}\Big[\frac{km+d}{p}\Big] \pmod{p^2}. \end{aligned}$$

Now putting all the above together we obtain the result.
Remark 4.1 In the case $d = 0$, Corollary 4.1 was first found by Lerch [Ler]. For a different proof of Lerch's result, see [S5].

Theorem 4.2. *Let p be a prime. Let f be a p-regular function. Suppose $m, n \in \mathbb{N}$ and $d, t \in \mathbb{Z}$ with $d, t \geq 0$. Then*

$$\sum_{r=0}^{n} \binom{n}{r} (-1)^r f(p^{m-1}rt + d) \equiv 0 \pmod{p^{mn}}.$$

Moreover, if $A_k = p^{-k} \sum_{r=0}^{k} \binom{k}{r} (-1)^r f(r)$, then

$$\begin{aligned} &\sum_{r=0}^{n} \binom{n}{r} (-1)^r f(p^{m-1}rt + d) \\ &\equiv \begin{cases} p^{mn} t^n A_n \pmod{p^{mn+1}} & \text{if } p > 2 \text{ or } m = 1, \\ 2^{mn} t^n \sum_{r=0}^{n} \binom{n}{r} A_{r+n} \pmod{2^{mn+1}} & \text{if } p = 2 \text{ and } m \geq 2. \end{cases} \end{aligned}$$

Proof. Since f is a p-regular function, we have $A_k \in \mathbb{Z}_p$ for $k \geq 0$. Set

$$a_0 = A_0 \quad \text{and} \quad a_i = (-1)^i \sum_{r=i}^{n} s(r, i) \frac{p^r}{r!} A_r \quad \text{for} \quad i = 1, 2, \ldots, n.$$

As $p^r/r! \in \mathbb{Z}_p$ and $A_r \in \mathbb{Z}_p$ we have $a_0, \ldots, a_n \in \mathbb{Z}_p$. From [S5, p. 197] we have

$$f(k) \equiv \sum_{i=0}^{n} a_i k^i \pmod{p^{n+1}} \quad \text{for} \quad k = 0, 1, 2, \ldots .$$

Thus applying (4.1) and (4.2) we see that

$$\begin{aligned} \sum_{r=0}^{n} \binom{n}{r} (-1)^r f(rt + d) &\equiv \sum_{r=0}^{n} \binom{n}{r} (-1)^r \sum_{i=0}^{n} a_i (rt + d)^i \\ &= \sum_{r=0}^{n} \binom{n}{r} (-1)^r (a_n t^n r^n + b_{n-1} r^{n-1} + \cdots + b_1 r + b_0) \\ &= a_n (-t)^n n! = (-1)^n s(n, n) \frac{p^n}{n!} A_n \cdot (-t)^n n! \\ &= p^n t^n A_n \pmod{p^{n+1}}, \end{aligned}$$

where $b_0, b_1, \ldots, b_{n-1} \in \mathbb{Z}_p$. Thus the result is true for $m = 1$.

Now assume $m \geq 2$. By the binomial inversion formula we have $f(k) = \sum_{s=0}^{k} \binom{k}{s} (-p)^s A_s$. Thus

$$\begin{aligned}
&\sum_{r=0}^{n} \binom{n}{r} (-1)^r f(p^{m-1} rt) \\
&= \sum_{r=0}^{n} \binom{n}{r} (-1)^r \sum_{k=0}^{p^{m-1}rt} \binom{p^{m-1}rt}{k} (-p)^k A_k \\
&= \sum_{k=0}^{p^{m-1}nt} (-p)^k A_k \sum_{r=0}^{n} \binom{n}{r} (-1)^r \binom{p^{m-1}rt}{k} \\
&= \sum_{k=n}^{p^{m-1}nt} (-p)^k A_k \cdot (-1)^n \frac{n!}{k!} \sum_{j=n}^{k} (-1)^{k-j} s(k,j) S(j,n) \big(p^{m-1}t\big)^j \quad \text{(by Lemma 4.1)} \\
&= \sum_{k=n}^{p^{m-1}nt} (-p)^n (-1)^k A_k \sum_{j=n}^{k} (-1)^{k-j} \frac{s(k,j)j!}{k!} p^{k-j} \cdot \frac{S(j,n)n!}{j!} p^{j-n} \cdot \big(p^{m-1}t\big)^j \\
&= A_n t^n p^{mn} + \sum_{k=n+1}^{p^{m-1}nt} (-p)^n (-1)^k A_k \Big(\frac{(-1)^{k-n} s(k,n) n!}{k!} p^{k-n} \cdot p^{(m-1)n} t^n \\
&\qquad + \sum_{j=n+1}^{k} \frac{(-1)^{k-j} s(k,j) j!}{k!} p^{k-j} \cdot \frac{S(j,n)n!}{j!} p^{j-n} \cdot (p^{m-1}t)^j \Big).
\end{aligned}$$

By Lemma 4.2, for $j, k, n \in \mathbb{N}$ we have

$$\frac{s(k,j)j!}{k!} p^{k-j} \in \mathbb{Z}_p \quad \text{and} \quad \frac{S(j,n)n!}{j!} p^{j-n} \in \mathbb{Z}_p.$$

Hence, by the above, Lemma 4.2 and the fact $(m-1)(n+1) + n \geq mn + 1$ we obtain

$$\begin{aligned}
&\sum_{r=0}^{n} \binom{n}{r} (-1)^r f(p^{m-1} rt) \\
&\equiv p^{mn} t^n \Big(A_n + \sum_{k=n+1}^{p^{m-1}nt} \frac{s(k,n)n!}{k!} p^{k-n} A_k \Big) \\
&\equiv \begin{cases} p^{mn} t^n A_n \pmod{p^{mn+1}} & \text{if } p > 2, \\ 2^{mn} t^n \sum_{k=n}^{2^{m-1}nt} \binom{n}{k-n} A_k = 2^{mn} t^n \sum_{r=0}^{n} \binom{n}{r} A_{r+n} \pmod{2^{mn+1}} & \text{if } p = 2. \end{cases}
\end{aligned}$$

Thus the result holds for $d = 0$.

Now suppose $g(r) = f(r + d)$. By the previous argument,

$$\sum_{r=0}^{n} \binom{n}{r} (-1)^r g(r) \equiv p^n A_n \pmod{p^{n+1}}.$$

Thus g is also a p-regular function. Note that

$$\sum_{r=0}^{n}\binom{n}{r}(-1)^r f(p^{m-1}rt+d)=\sum_{r=0}^{n}\binom{n}{r}(-1)^r g(p^{m-1}rt).$$

By the above we see that the result is also true for $d>0$. The proof is now complete.

Theorem 4.3. *Let p be a prime, $k,m,n,t\in\mathbb{N}$ and $d\in\{0,1,2,\dots\}$. Let f be a p-regular function. Then*

$$f(ktp^{m-1}+d)\equiv\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}f(rtp^{m-1}+d)\pmod{p^{mn}}.$$

Moreover, setting $A_s=p^{-s}\sum_{r=0}^{s}\binom{s}{r}(-1)^r f(r)$ we then have

$$\begin{aligned}&f(ktp^{m-1}+d)-\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}f(rtp^{m-1}+d)\\&\equiv\begin{cases}p^{mn}\binom{k}{n}(-t)^nA_n\pmod{p^{mn+1}} & \text{if } p>2 \text{ or } m=1,\\ 2^{mn}\binom{k}{n}(-t)^n\sum_{r=0}^{n}\binom{n}{r}A_{r+n}\pmod{2^{mn+1}} & \text{if } p=2 \text{ and } m\ge 2.\end{cases}\end{aligned}$$

Proof. From [S4, Lemma 2.1] we know that for any function F,

$$\begin{aligned}F(k)=&\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}F(r)\\&+\sum_{r=n}^{k}\binom{k}{r}(-1)^r\sum_{s=0}^{r}\binom{r}{s}(-1)^sF(s),\end{aligned}\tag{4.5}$$

where the second sum vanishes when $k<n$.

Now taking $F(k)=f(ktp^{m-1}+d)$ we obtain

$$\begin{aligned}f(ktp^{m-1}+d)=&\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}f(rtp^{m-1}+d)\\&+\sum_{r=n}^{k}\binom{k}{r}(-1)^r\sum_{s=0}^{r}\binom{r}{s}(-1)^sf(stp^{m-1}+d).\end{aligned}$$

By Theorem 4.2 we have

$$\begin{aligned}&\sum_{r=n}^{k}\binom{k}{r}(-1)^r\sum_{s=0}^{r}\binom{r}{s}(-1)^sf(stp^{m-1}+d)\\&\equiv(-1)^n\binom{k}{n}\sum_{s=0}^{n}\binom{n}{s}(-1)^sf(stp^{m-1}+d)\\&\equiv\begin{cases}\binom{k}{n}p^{mn}(-t)^nA_n\pmod{p^{mn+1}} & \text{if } p>2 \text{ or } m=1,\\ \binom{k}{n}2^{mn}(-t)^n\sum_{r=0}^{n}\binom{n}{r}A_{r+n}\pmod{2^{mn+1}} & \text{if } p=2 \text{ and } m\ge 2.\end{cases}\end{aligned}$$

Now combining the above we prove the theorem.

Putting $n=1,2,3$ and $d=0$ in Theorem 4.3 we deduce the following result.

Corollary 4.2. *Let p be a prime, $k, m, t \in \mathbb{N}$. Let f be a p-regular function. Then*

(i) ([S5, Corollary 2.1]) $f(kp^{m-1}) \equiv f(0) \pmod{p^m}$.

(ii) $f(ktp^{m-1}) \equiv kf(tp^{m-1}) - (k-1)f(0) \pmod{p^{2m}}$.

(iii) *We have*

$$\begin{aligned} f(ktp^{m-1}) \equiv &\frac{k(k-1)}{2} f(2tp^{m-1}) - k(k-2)f(tp^{m-1}) \\ &+ \frac{(k-1)(k-2)}{2} f(0) \pmod{p^{3m}}. \end{aligned}$$

(iv) *We have*

$$\begin{aligned} &f(kp^{m-1}) \\ &\equiv \begin{cases} f(0) - k(f(0)-f(1))p^{m-1} \pmod{p^{m+1}} & \text{if } p > 2 \text{ or } m = 1, \\ f(0) - 2^{m-2}k(f(2) - 4f(1) + 3f(0)) \pmod{2^{m+1}} & \text{if } p = 2 \text{ and } m \ge 2. \end{cases} \end{aligned}$$

Theorem 4.4. *Let p be a prime and let f be a p-regular function. Let $n \in \mathbb{N}$.*

(i) *For $d, x \in \mathbb{Z}_p$ and $m \in \{0, 1, \dots, n-1\}$ we have*

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \binom{kx+d}{m} f(k) \equiv 0 \pmod{p^{n-m}}.$$

(ii) *We have*

$$\sum_{k=1}^{n} \binom{n}{k} (-1)^k f(k-1) \equiv -f(p^{n-1}-1) \pmod{p^n}.$$

Proof. From [S5, Theorem 2.1] we know that there are $a_0, a_1, \dots, a_{n-m-1} \in \mathbb{Z}$ such that

$$f(k) \equiv a_{n-m-1}k^{n-m-1} + \dots + a_1 k + a_0 \pmod{p^{n-m}} \quad \text{for} \quad k = 0, 1, 2, \dots$$

Thus applying Lemma 4.1 and (4.1) we have

$$\begin{aligned} &\sum_{k=0}^{n} \binom{n}{k} (-1)^k \binom{kx+d}{m} f(k) \\ &\equiv \sum_{k=0}^{n} \binom{n}{k} (-1)^k \binom{kx+d}{m} \sum_{i=0}^{n-m-1} a_i k^i \\ &= \sum_{i=0}^{n-m-1} a_i \sum_{k=0}^{n} \binom{n}{k} (-1)^k \binom{kx+d}{m} k^i = 0 \pmod{p^{n-m}}. \end{aligned}$$

This proves (i).

Now we consider (ii). By [S5, Theorem 2.1] there are $a_0, a_1, \dots, a_{n-1} \in \mathbb{Z}_p$ such that $s!a_s/p^s \in \mathbb{Z}_p$ $(s = 0, 1, \dots, n-1)$ and

$$f(k) \equiv a_{n-1}k^{n-1} + \cdots + a_1 k + a_0 \pmod{p^n} \quad \text{for} \quad k = 0, 1, 2, \dots$$

Note that $p^{s-1}/s! \in \mathbb{Z}_p$ for $s \in \mathbb{N}$. We then have $a_1 \equiv \cdots \equiv a_{n-1} \equiv 0 \pmod p$. Let

$$a_{n-1}(k-1)^{n-1} + \cdots + a_1(k-1) + a_0 = b_{n-1}k^{n-1} + \cdots + b_1 k + b_0.$$

Then clearly $b_1 \equiv \cdots \equiv b_{n-1} \equiv 0 \pmod p$ and

$$f(k-1) \equiv b_{n-1}k^{n-1} + \cdots + b_1 k + b_0 \pmod{p^n} \quad \text{for} \quad k = 1, 2, 3, \dots$$

Thus

$$f(p^{n-1}-1) \equiv b_{n-1}(p^{n-1})^{n-1} + \cdots + b_1 p^{n-1} + b_0 \equiv b_0 \pmod{p^n}.$$

Hence, applying (4.1) we have

$$\begin{aligned}\sum_{k=1}^{n}\binom{n}{k}(-1)^k f(k-1) &\equiv \sum_{k=1}^{n}\binom{n}{k}(-1)^k(b_{n-1}k^{n-1} + \cdots + b_1 k + b_0)\\ &= \sum_{i=1}^{n-1} b_i \sum_{k=0}^{n}\binom{n}{k}(-1)^k k^i + b_0 \sum_{k=1}^{n}\binom{n}{k}(-1)^k\\ &= -b_0 \equiv -f(p^{n-1}-1) \pmod{p^n}.\end{aligned}$$

So the theorem is proved.

5. Congruences for $pB_{k\varphi(p^m)+b}(x)$ and $pB_{k\varphi(p^m)+b,\chi} \pmod{p^{mn}}$.

For given prime p and $t \in \mathbb{Z}_p$ we recall that $\langle t\rangle_p$ denotes the least nonnegative residue of t modulo p.

Theorem 5.1. *Let p be a prime, and $k, m, n, t, b \in \mathbb{Z}$ with $m, n \ge 1$ and $k, b, t \ge 0$. Let $x \in \mathbb{Z}_p$ and $x' = (x + \langle -x\rangle_p)/p$. Then*

$$\begin{aligned}&pB_{kt\varphi(p^m)+b}(x) - p^{kt\varphi(p^m)+b}B_{kt\varphi(p^m)+b}(x')\\ &- \sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}\Big(pB_{rt\varphi(p^m)+b}(x) - p^{rt\varphi(p^m)+b}B_{rt\varphi(p^m)+b}(x')\Big)\\ &\equiv \begin{cases} \delta(b,n,p)\binom{k}{n}(-t)^n p^{mn-1} \pmod{p^{mn}} & \text{if } p > 2 \text{ or } m = 1,\\ 0 \pmod{2^{mn}} & \text{if } p = 2 \text{ and } m \ge 2,\end{cases}\end{aligned}$$

where

$$\delta(b,n,p) = \begin{cases} 1 & \text{if } p = 2 \text{ and } n \in \{1, 2, 4, 6, \dots\}\\ & \quad \text{or if } p > 2,\ p-1 \mid b \text{ and } p-1 \mid n,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. From [S4, Theorem 3.1] we know that

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\Big(pB_{k(p-1)+b}(x)-p^{k(p-1)+b}B_{k(p-1)+b}(x')\Big)\equiv p^{n-1}\delta(b,n,p)\pmod{p^n}.$$

Set $f(k)=p\Big(pB_{k(p-1)+b}(x)-p^{k(p-1)+b}B_{k(p-1)+b}(x')\Big)$. Then $\sum_{k=0}^{n}\binom{n}{k}(-1)^kf(k)\equiv\delta(b,n,p)p^n\pmod{p^{n+1}}$. Thus f is a p-regular function. Hence appealing to Theorem 4.3 we have

$$\begin{aligned}&f(ktp^{m-1})-\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}f(rtp^{m-1})\\&\equiv\begin{cases}p^{mn}\binom{k}{n}(-t)^n\delta(b,n,p)\pmod{p^{mn+1}}&\text{if }p>2\text{ or }m=1,\\2^{mn}\binom{k}{n}(-t)^n\sum_{r=0}^{n}\binom{n}{r}\delta(b,n+r,2)\pmod{2^{mn+1}}&\text{if }p=2\text{ and }m\ge 2.\end{cases}\end{aligned}$$

Note that

$$\delta(b,n+r,2)=\begin{cases}1&\text{if }n+r\in\{1,2,4,6,\dots\},\\0&\text{if }n+r\in\{3,5,7,\dots\}.\end{cases}$$

We then have

$$\begin{aligned}&\sum_{r=0}^{n}\binom{n}{r}\delta(b,n+r,2)\\&=\begin{cases}\delta(b,1,2)+\delta(b,2,2)=1+1\equiv 0\pmod 2&\text{if }n=1,\\\sum\limits_{\substack{r=0\\2\mid n+r}}^{n}\binom{n}{r}=2^{n-1}\equiv 0\pmod 2&\text{if }n>1.\end{cases}\end{aligned}$$

Thus

$$\begin{aligned}&\frac{f(ktp^{m-1})}{p}-\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}\frac{f(rtp^{m-1})}{p}\\&\equiv\begin{cases}p^{mn-1}\binom{k}{n}(-t)^n\delta(b,n,p)\pmod{p^{mn}}&\text{if }p>2\text{ or }m=1,\\0\pmod{2^{mn}}&\text{if }p=2\text{ and }m\ge 2.\end{cases}\end{aligned}$$

This is the result.

Corollary 5.1. *Let p be a prime, and $k,m,b\in\mathbb{Z}$ with $k,m\ge 1$ and $b\ge 0$. Let $x\in\mathbb{Z}_p$ and $x'=(x+\langle -x\rangle_p)/p$. Suppose $p>2$ or $m>1$. Then*

$$pB_{k\varphi(p^m)+b}(x)\equiv\begin{cases}3\pmod 4&\text{if }p=m=2,\ k=1\text{ and }b=0,\\pB_b(x)-p^bB_b(x')\pmod{p^m}&\text{otherwise.}\end{cases}$$

Proof. Putting $n=t=1$ in Theorem 5.1 we see that

$$pB_{k\varphi(p^m)+b}(x)-p^{k\varphi(p^m)+b}B_{k\varphi(p^m)+b}(x')\equiv pB_b(x)-p^bB_b(x')\pmod{p^m}.$$

If $p=m=2$, $k=1$ and $b=0$, then $pB_{k\varphi(p^m)+b}(x)=2B_2(x)=2(x^2-x+\frac{1}{6})\equiv 3 \pmod 4$. Otherwise, we have $k\varphi(p^m)+b\ge m+1$ and so $p^{k\varphi(p^m)+b}B_{k\varphi(p^m)+b}(x')\equiv 0 \pmod{p^m}$. Thus the result follows from the above.

In the case $p>2$, Corollary 5.1 has been proved by the author in [S4].

Let χ be a primitive Dirichlet character of conductor m. The generalized Bernoulli number $B_{n,\chi}$ is defined by

$$\sum_{r=1}^{m}\frac{\chi(r)t\mathrm{e}^{rt}}{\mathrm{e}^{mt}-1}=\sum_{n=0}^{\infty}B_{n,\chi}\frac{t^n}{n!}.$$

Let χ_0 be the trivial character. It is well known that (see [W])

$$B_{1,\chi_0}=\frac{1}{2},\ B_{n,\chi_0}=B_n\ (n\ne 1)\quad\text{and}\quad B_{n,\chi}=m^{n-1}\sum_{r=1}^{m}\chi(r)B_n\Big(\frac{r}{m}\Big).$$

If χ is nontrivial and $n\in\mathbb{N}$, then clearly $\sum_{r=1}^{m}\chi(r)=0$ and so

$$\frac{B_{n,\chi}}{n}=m^{n-1}\sum_{r=1}^{m}\chi(r)\Big(\frac{B_n(\frac{r}{m})-B_n}{n}+\frac{B_n}{n}\Big)=m^{n-1}\sum_{r=1}^{m}\chi(r)\frac{B_n(\frac{r}{m})-B_n}{n}.$$

When p is a prime with $p\nmid m$, by [S4, Lemma 2.3] we have $(B_n(\frac{r}{m})-B_n)/n\in\mathbb{Z}_p$. Thus $B_{n,\chi}/n$ is congruent to an algebraic integer modulo p.

Lemma 5.1. *Let p be a prime and let b be a nonnegative integer.*

(i) ([S5, Theorem 3.2], [Y2]) *If $p-1\nmid b$, $x\in\mathbb{Z}_p$ and $x'=(x+\langle -x\rangle_p)/p$, then $f(k)=(B_{k(p-1)+b}(x)-p^{k(p-1)+b-1}B_{k(p-1)+b}(x'))/(k(p-1)+b)$ is a $p-$regular function.*

(ii) ([S5, (3.1), Theorem 3.1 and Remark 3.1]) *If $a,b\in\mathbb{N}$ and $p\nmid a$, then $f(k)=(1-p^{k(p-1)+b-1})(a^{k(p-1)+b}-1)B_{k(p-1)+b}/(k(p-1)+b)$ is a p-regular function.*

(iii) ([Y3, Theorem 4.2], [Y1, p. 216], [F], [S5, Lemma 8.1(a)]) *If $b,m\in\mathbb{N}$, $p\nmid m$ and χ is a nontrivial primitive Dirichlet character of conductor m, then $f(k)=(1-\chi(p)\ p^{k(p-1)+b-1})B_{k(p-1)+b,\chi}/(k(p-1)+b)$ is a $p-$regular function.*

(iv) ([S5, Lemma 8.1(b)]) *If $m\in\mathbb{N}$, $p\nmid m$ and χ is a nontrivial Dirichlet character of conductor m, then $f(k)=(1-\chi(p)p^{k(p-1)+b-1})pB_{k(p-1)+b,\chi}$ is a $p-$regular function.*

From Lemma 5.1 and Theorem 4.3 we deduce the following theorem.

Theorem 5.2. *Let p be a prime, $k,n,s,t\in\mathbb{N}$ and $b\in\{0,1,2,\dots\}$.*

(i) *If $p-1\nmid b$, $x\in\mathbb{Z}_p$ and $x'=(x+\langle -x\rangle_p)/p$, then*

$$\begin{aligned}&\frac{B_{ktp^{s-1}(p-1)+b}(x)-p^{ktp^{s-1}(p-1)+b-1}B_{ktp^{s-1}(p-1)+b}(x')}{ktp^{s-1}(p-1)+b}\\&\equiv\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}\\&\quad\times\frac{B_{rtp^{s-1}(p-1)+b}(x)-p^{rtp^{s-1}(p-1)+b-1}B_{rtp^{s-1}(p-1)+b}(x')}{rtp^{s-1}(p-1)+b}\pmod{p^{sn}}.\end{aligned}$$

(ii) *If $a, b \in \mathbb{N}$ and $p \nmid a$, then*

$$\begin{aligned}&\big(1-p^{ktp^{s-1}(p-1)+b-1}\big)\big(a^{ktp^{s-1}(p-1)+b}-1\big)\frac{B_{ktp^{s-1}(p-1)+b}}{ktp^{s-1}(p-1)+b}\\ &\equiv\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}\big(1-p^{rtp^{s-1}(p-1)+b-1}\big)\\ &\quad\times\big(a^{rtp^{s-1}(p-1)+b}-1\big)\frac{B_{rtp^{s-1}(p-1)+b}}{rtp^{s-1}(p-1)+b}\pmod{p^{sn}}.\end{aligned}$$

(iii) *If $b, m \in \mathbb{N}$, $p \nmid m$ and χ is a nontrivial primitive Dirichlet character of conductor m, then*

$$\begin{aligned}&\frac{(1-\chi(p)p^{ktp^{s-1}(p-1)+b-1})B_{ktp^{s-1}(p-1)+b,\chi}}{ktp^{s-1}(p-1)+b}\\ &\equiv\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}\\ &\quad\times\frac{(1-\chi(p)p^{rtp^{s-1}(p-1)+b-1})B_{rtp^{s-1}(p-1)+b,\chi}}{rtp^{s-1}(p-1)+b}\pmod{p^{sn}}.\end{aligned}$$

(iv) *If $m \in \mathbb{N}$, $p \nmid m$ and χ is a nontrivial Dirichlet character of conductor m, then*

$$\begin{aligned}&\big(1-\chi(p)p^{ktp^{s-1}(p-1)+b-1}\big)pB_{ktp^{s-1}(p-1)+b,\chi}\\ &\equiv\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}\\ &\quad\times\big(1-\chi(p)p^{rtp^{s-1}(p-1)+b-1}\big)pB_{rtp^{s-1}(p-1)+b,\chi}\pmod{p^{sn}}.\end{aligned}$$

Remark 5.1 Theorem 5.2 can be viewed as generalizations of some congruences in [S5]. In the case $n = 1$, Theorem 5.2(i) was given by Eie and Ong [EO], and independently by the author in [S5, p. 204]. In the case $s = t = 1$, Theorem 5.2(i) was announced by the author in [S4] and proved in [S5], and Theorem 5.2(iii) (in the case $p - 1 \nmid b$) and Theorem 5.2(iv) were also given in [S5]. When $n = 1$, Theorem 5.2(iii) was given in [W, p. 141].

Combining Lemma 5.1 and Corollary 4.2(iv) we obtain the following result.

Theorem 5.3. *Let p be an odd prime, $k, s \in \mathbb{N}$ and $b \in \{0, 1, 2, \dots\}$.*

(i) *If $p - 1 \nmid b$, $x \in \mathbb{Z}_p$ and $x' = (x + \langle -x\rangle_p)/p$, then*

$$\frac{B_{k\varphi(p^s)+b}(x)}{k\varphi(p^s)+b}\equiv(1-kp^{s-1})\frac{B_b(x)-p^{b-1}B_b(x')}{b}+kp^{s-1}\frac{B_{p-1+b}(x)}{p-1+b}\pmod{p^{s+1}}.$$

(ii) *If $b, m \in \mathbb{N}$, $p \nmid m$ and χ is a nontrivial primitive Dirichlet character of conductor m, then*

$$\frac{B_{k\varphi(p^s)+b,\chi}}{k\varphi(p^s)+b} \equiv (1-kp^{s-1})\big(1-\chi(p)p^{b-1}\big)\frac{B_{b,\chi}}{b} + kp^{s-1}\frac{B_{p-1+b,\chi}}{p-1+b} \pmod{p^{s+1}}.$$

(iii) *If $m \in \mathbb{N}$, $p \nmid m$ and χ is a nontrivial Dirichlet character of conductor m, then*

$$\begin{aligned}&(1-\chi(p)p^{k\varphi(p^s)+b-1})pB_{k\varphi(p^s)+b,\chi}\\ &\equiv (1-kp^{s-1})\big(1-\chi(p)p^{b-1}\big)pB_{b,\chi}\\ &\quad + kp^{s-1}\big(1-\chi(p)p^{p-2+b}\big)pB_{p-1+b,\chi} \pmod{p^{s+1}}.\end{aligned}$$

Corollary 5.2. *Let p be an odd prime and $k, s, b \in \mathbb{N}$ with $2 \mid b$ and $p-1 \nmid b$. Then*

$$\frac{B_{k\varphi(p^s)+b}}{k\varphi(p^s)+b} \equiv (1-kp^{s-1})(1-p^{b-1})\frac{B_b}{b} + kp^{s-1}\frac{B_{p-1+b}}{p-1+b} \pmod{p^{s+1}}.$$

Theorem 5.4. *Let p be a prime, $a, n \in \mathbb{N}$ and $p \nmid a$.*
(i) *There are integers $b_0, b_1, \cdots, b_{n-1}$ such that*

$$\begin{aligned}&\big(1-p^{k(p-1)-1}\big)\big(a^{k(p-1)}-1\big)\frac{B_{k(p-1)}}{k(p-1)}\\ &\equiv b_{n-1}k^{n-1}+\cdots+b_1k+b_0 \pmod{p^n} \quad \textit{for} \quad k=1,2,3,\ldots\end{aligned}$$

(ii) *If $p > 2$ or $n > 2$, then*

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^k(1-p^{k(p-1)-1})(a^{k(p-1)}-1)\frac{B_{k(p-1)}}{k(p-1)} \equiv \frac{1-a^{\varphi(p^n)}}{p^n} \pmod{p^n}.$$

Proof. Suppose $b \in \mathbb{N}$. From Lemma 5.1(ii) we know that

$$f(k) = \left(1-p^{k(p-1)+b-1}\right)\left(a^{k(p-1)+b}-1\right)\frac{B_{k(p-1)+b}}{k(p-1)+b}$$

is a p-regular function. Hence taking $b = p-1$ and applying [S5, Theorem 2.1] we know that there exist integers $a_0, a_1, \ldots, a_{n-1}$ such that

$$\begin{aligned}&\big(1-p^{(k+1)(p-1)-1}\big)\big(a^{(k+1)(p-1)}-1\big)\frac{B_{(k+1)(p-1)}}{(k+1)(p-1)}\\ &\equiv a_{n-1}k^{n-1}+\cdots+a_1k+a_0 \pmod{p^n} \quad \text{for} \quad k=0,1,2,\ldots\end{aligned}$$

That is,

$$\begin{aligned}&\big(1-p^{k(p-1)-1}\big)\big(a^{k(p-1)}-1\big)\frac{B_{k(p-1)}}{k(p-1)}\\ &\equiv a_{n-1}(k-1)^{n-1}+\cdots+a_1(k-1)+a_0 \pmod{p^n} \quad \text{for} \quad k=1,2,3,\ldots\end{aligned}$$

On setting

$$a_{n-1}(k-1)^{n-1}+\cdots+a_1(k-1)+a_0=b_{n-1}k^{n-1}+\cdots+b_1k+b_0$$

we obtain (i).

Now we consider (ii). Suppose $p>2$ or $n>2$. Since $f(k)$ is a p-regular function, by Theorem 4.4(ii) we have

$$\begin{aligned}&\sum_{k=1}^{n}\binom{n}{k}(-1)^k(1-p^{(k-1)(p-1)+b-1})(a^{(k-1)(p-1)+b}-1)\frac{B_{(k-1)(p-1)+b}}{(k-1)(p-1)+b}\\&\equiv -(1-p^{(p^{n-1}-1)(p-1)+b-1})(a^{(p^{n-1}-1)(p-1)+b}-1)\frac{B_{(p^{n-1}-1)(p-1)+b}}{(p^{n-1}-1)(p-1)+b}\pmod{p^n}.\end{aligned}$$

Substituting b by $p-1+b$ we see that for $b\ge 0$,

$$\begin{aligned}&\sum_{k=1}^{n}\binom{n}{k}(-1)^k(1-p^{k(p-1)+b-1})(a^{k(p-1)+b}-1)\frac{B_{k(p-1)+b}}{k(p-1)+b}\\&\equiv -(1-p^{\varphi(p^n)+b-1})(a^{\varphi(p^n)+b}-1)\frac{B_{\varphi(p^n)+b}}{\varphi(p^n)+b}\pmod{p^n}.\end{aligned}\tag{5.1}$$

By Corollary 5.1 we have $pB_{\varphi(p^n)}\equiv p-1\pmod{p^n}$. Thus taking $b=0$ in (5.1) and noting that $\varphi(p^n)\ge n+1$ we obtain

$$\begin{aligned}&\sum_{k=1}^{n}\binom{n}{k}(-1)^k(1-p^{k(p-1)-1})(a^{k(p-1)}-1)\frac{B_{k(p-1)}}{k(p-1)}\\&\equiv -(1-p^{\varphi(p^n)-1})(a^{\varphi(p^n)}-1)\frac{B_{\varphi(p^n)}}{\varphi(p^n)}\\&=-(1-p^{\varphi(p^n)-1})\frac{a^{\varphi(p^n)}-1}{p^n}\cdot\frac{pB_{\varphi(p^n)}}{p-1}\equiv-\frac{a^{\varphi(p^n)}-1}{p^n}\pmod{p^n}.\end{aligned}$$

This completes the proof of the theorem.

6. Congruences for $\sum_{k=0}^{n}\binom{n}{k}(-1)^kpB_{k(p-1)+b}(x)\pmod{p^{n+1}}$.

For $a\in\mathbb{N}$ and $b\in\mathbb{Z}$ we define $\chi(a\mid b)=1$ or 0 according as $a\mid b$ or $a\nmid b$.

Lemma 6.1. *Let p be an odd prime and $n\in\mathbb{N}$. Then*

$$\sum_{\substack{s=1\\ s\equiv n+1\ (\mathrm{mod}\ p-1)}}^{n}\binom{n}{s}\equiv-\chi(p-1\mid n)\pmod{p}.$$

Proof. Let $n_0\in\{1,2,\ldots,p-1\}$ be such that $n\equiv n_0\pmod{p-1}$. Since Glaisher (see [D]) it is well known that

$$\sum_{\substack{s=0\\ s\equiv r\ (\mathrm{mod}\ p-1)}}^{n}\binom{n}{s}\equiv\sum_{\substack{s=0\\ s\equiv r\ (\mathrm{mod}\ p-1)}}^{n_0}\binom{n_0}{s}\pmod{p}\quad\text{for}\quad r\in\mathbb{Z}.$$

From [S1] we know that

$$\sum_{\substack{s=0\\ s\equiv r \pmod{p-1}}}^{n} \binom{n}{s} = \sum_{\substack{s=0\\ s\equiv n-r \pmod{p-1}}}^{n} \binom{n}{s}.$$

Thus

$$\sum_{\substack{s=0\\ s\equiv n+1 \pmod{p-1}}}^{n} \binom{n}{s} = \sum_{\substack{s=0\\ s\equiv -1 \pmod{p-1}}}^{n} \binom{n}{s} \equiv \sum_{\substack{s=0\\ s\equiv p-2 \pmod{p-1}}}^{n_0} \binom{n_0}{s}$$
$$= \begin{cases} p-1\equiv -1 \pmod{p} & \text{if } n_0 = p-1,\\ 1 \pmod{p} & \text{if } n_0 = p-2,\\ 0 \pmod{p} & \text{if } n_0 < p-2. \end{cases}$$

Hence

$$\sum_{\substack{s=1\\ s\equiv n+1 \pmod{p-1}}}^{n} \binom{n}{s} = \sum_{\substack{s=0\\ s\equiv n+1 \pmod{p-1}}}^{n} \binom{n}{s} - \chi(p-1 \mid n+1) \equiv -\chi(p-1 \mid n) \pmod{p}.$$

This proves the lemma.

Proposition 6.1. *Let p be an odd prime, $n \in \mathbb{N}$ and $x \in \mathbb{Z}_p$. Let b be a nonnegative integer. Then*

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \Big(pB_{k(p-1)+b}(x) - p^{k(p-1)+b} B_{k(p-1)+b}\Big(\frac{x+\langle -x\rangle_p}{p}\Big)\Big)$$
$$\equiv \sum_{\substack{j=0\\ j\neq \langle -x\rangle_p}}^{p-1} (x+j)^{b-n} p^n B_n\Big(\frac{(x+j)^p-(x+j)}{p(p-1)}\Big) + p^n \Delta(b,n,p) \pmod{p^{n+1}},$$

where

$$\Delta(b,n,p) = \begin{cases} (n-b)T - n & \text{if } p-1 \mid b \text{ and } p-1 \mid n,\\ (n-b)T & \text{if } p-1 \nmid b \text{ and } p-1 \mid n,\\ b-n & \text{if } p-1 \mid b \text{ and } p-1 \mid n+1,\\ 0 & \text{otherwise} \end{cases}$$

and

$$T = \sum_{\substack{j=0\\ j\neq \langle -x\rangle_p}}^{p-1} \frac{(x+j)^{p-1+b} - (x+j)^b}{p}.$$

Proof. Let

$$S_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \Big(pB_{k(p-1)+b}(x) - p^{k(p-1)+b} B_{k(p-1)+b}\Big(\frac{x+\langle -x\rangle_p}{p}\Big)\Big).$$

From [S4, p.157] we know that

$$S_n=\sum_{r=0}^{n(p-1)+b}p^rB_r\sum_{\substack{j=0\\ j\ne\langle -x\rangle_p}}^{p-1}\sum_{k=0}^{n}\binom nk(-1)^k\binom{k(p-1)+b}{r}(x+j)^{k(p-1)+b-r}.$$

By [S5, p.199] we know that for any functions f and g we have

$$\begin{aligned}&\sum_{k=0}^{n}\binom nk(-1)^kf(k)g(k)\\ &=\sum_{s=0}^{n}\binom ns\Big(\sum_{i=0}^{n-s}\binom{n-s}{i}(-1)^if(i+s)\Big)\sum_{j=0}^{s}\binom sj(-1)^jg(j).\end{aligned}\tag{6.1}$$

Now taking $f(k)=\binom{k(p-1)+b}{r}$ and $g(k)=a^{k(p-1)+b-r}$ $(a\ne 0)$ in (6.1) we obtain

$$\begin{aligned}&\sum_{k=0}^{n}\binom nk(-1)^k\binom{k(p-1)+b}{r}a^{k(p-1)+b-r}\\ &=\sum_{s=0}^{n}\binom ns\Big(\sum_{i=0}^{n-s}\binom{n-s}{i}(-1)^i\binom{(i+s)(p-1)+b}{r}\Big)\sum_{j=0}^{s}\binom sj(-1)^ja^{j(p-1)+b-r}\\ &=\sum_{s=0}^{n}\binom ns a^{b-r}(1-a^{p-1})^s\sum_{i=0}^{n-s}\binom{n-s}{i}(-1)^i\binom{i(p-1)+s(p-1)+b}{r}.\end{aligned}$$

Thus applying the above and Lemma 4.1 we have

$$\begin{aligned}S_n&=\sum_{r=0}^{n(p-1)+b}p^rB_r\sum_{\substack{j=0\\ j\ne\langle -x\rangle_p}}^{p-1}\sum_{s=0}^{n}\binom ns(x+j)^{b-r}\big(1-(x+j)^{p-1}\big)^s\\ &\qquad\times\sum_{i=0}^{n-s}\binom{n-s}{i}(-1)^i\binom{i(p-1)+s(p-1)+b}{r}\\ &=\sum_{\substack{j=0\\ j\ne\langle -x\rangle_p}}^{p-1}\sum_{s=0}^{n}\binom ns\Big(\frac{1-(x+j)^{p-1}}{p}\Big)^s\sum_{r=n-s}^{n(p-1)+b}p^{r+s}B_r\cdot(x+j)^{b-r}\\ &\qquad\times\sum_{i=0}^{n-s}\binom{n-s}{i}(-1)^i\binom{i(p-1)+s(p-1)+b}{r}.\end{aligned}$$

Since $pB_r\in\mathbb{Z}_p$ and so $p^{r+s}B_r\equiv 0\pmod{p^{n+1}}$ for $r\ge n-s+2$, by Theorem 4.1 we have

$$\sum_{r=n-s}^{n(p-1)+b}(x+j)^{b-r}p^{r+s}B_r\sum_{i=0}^{n-s}\binom{n-s}{i}(-1)^i\binom{i(p-1)+s(p-1)+b}{r}$$

$$\equiv (x+j)^{b-(n-s)}p^nB_{n-s}\sum_{i=0}^{n-s}\binom{n-s}{i}(-1)^i\binom{i(p-1)+s(p-1)+b}{n-s}$$

$$+ (x+j)^{b-(n-s+1)}p^{n+1}B_{n-s+1}\sum_{i=0}^{n-s}\binom{n-s}{i}(-1)^i\binom{i(p-1)+s(p-1)+b}{n-s+1}$$

$$= (x+j)^{b-(n-s)}p^nB_{n-s}\cdot(1-p)^{n-s}+(x+j)^{b-(n-s+1)}p^{n+1}B_{n-s+1}$$

$$\times (s(p-1)+b+(n-s)(p-2)/2)(1-p)^{n-s}$$

$$\equiv (x+j)^{b-(n-s)}(1-p)^{n-s}p^nB_{n-s}$$

$$+ (x+j)^{b-(n-s+1)}(b-n)p^{n+1}B_{n-s+1} \pmod{p^{n+1}}.$$

Thus,

$$S_n \equiv \sum_{\substack{j=0\\ j\neq\langle -x\rangle_p}}^{p-1}\sum_{s=0}^{n}\binom{n}{s}\Big(\frac{1-(x+j)^{p-1}}{p}\Big)^s\Big((x+j)^{b-n+s}(1-p)^{n-s}p^nB_{n-s}$$

$$+ (x+j)^{b-n+s-1}(b-n)p^{n+1}B_{n-s+1}\Big)$$

$$= \sum_{\substack{j=0\\ j\neq\langle -x\rangle_p}}^{p-1}(x+j)^{b-n}(1-p)^np^n\sum_{s=0}^{n}\binom{n}{s}\Big(\frac{1-(x+j)^{p-1}}{p}\cdot\frac{x+j}{1-p}\Big)^sB_{n-s}$$

$$+ \sum_{\substack{j=0\\ j\neq\langle -x\rangle_p}}^{p-1}\sum_{s=0}^{n}\binom{n}{s}\Big(\frac{1-(x+j)^{p-1}}{p}\Big)^s(x+j)^{b-n+s-1}(b-n)p^{n+1}B_{n-s+1}$$

$$\equiv \sum_{\substack{j=0\\ j\neq\langle -x\rangle_p}}^{p-1}(x+j)^{b-n}(1-p)^np^nB_n(x_j)+\sum_{\substack{j=0\\ j\neq\langle -x\rangle_p}}^{p-1}\sum_{\substack{s=0\\ p-1\mid n-s+1}}^{n}\binom{n}{s}$$

$$\times\Big(\frac{1-(x+j)^{p-1}}{p}\Big)^s(x+j)^{b-n+s-1}(n-b)p^n \pmod{p^{n+1}},$$

where

$$x_j=\frac{(x+j)^p-(x+j)}{p(p-1)}.$$

In the last step we use the facts

$$B_n(t)=\sum_{s=0}^{n}\binom{n}{s}t^sB_{n-s}\quad\text{and}\quad pB_k\equiv-\chi(p-1\mid k)\pmod{p}\ (k\geq 1).$$

For $a \in \mathbb{Z}$, using Lemma 6.1 and Fermat's little theorem we see that

$$\begin{aligned}\sum_{\substack{s=0\\ s\equiv n+1 \pmod{p-1}}}^{n} \binom{n}{s} a^s &= \sum_{\substack{s=1\\ s\equiv n+1 \pmod{p-1}}}^{n} \binom{n}{s} a^s + \chi(p-1 \mid n+1)\\ &\equiv a^{n+1} \sum_{\substack{s=1\\ s\equiv n+1 \pmod{p-1}}}^{n} \binom{n}{s} + \chi(p-1 \mid n+1)\\ &\equiv -\chi(p-1 \mid n)a^{n+1} + \chi(p-1 \mid n+1)\\ &= \begin{cases} -a^{n+1} \equiv -a \pmod{p} & \text{if } p-1 \mid n,\\ 1 \pmod{p} & \text{if } p-1 \mid n+1,\\ 0 \pmod{p} & \text{if } p-1 \nmid n \text{ and } p-1 \nmid n+1.\end{cases}\end{aligned}$$

We also note that (see [S5, (5.1)])

$$\sum_{\substack{j=0\\ j\neq \langle -x\rangle_p}}^{p-1} (x+j)^b \equiv \sum_{r=1}^{p-1} r^b \equiv -\chi(p-1 \mid b) \pmod{p}. \tag{6.2}$$

Thus

$$\begin{aligned}&\sum_{\substack{j=0\\ j\neq \langle -x\rangle_p}}^{p-1} \sum_{\substack{s=0\\ p-1\mid n-s+1}}^{n} \binom{n}{s} \Big(\frac{1-(x+j)^{p-1}}{p}\Big)^s (x+j)^{b-n+s-1}(n-b)p^n\\ &\equiv p^n(n-b) \sum_{\substack{j=0\\ j\neq \langle -x\rangle_p}}^{p-1} (x+j)^b \sum_{\substack{s=0\\ s\equiv n+1 \pmod{p-1}}}^{n} \binom{n}{s} \Big(\frac{1-(x+j)^{p-1}}{p}\Big)^s\\ &\equiv \begin{cases} p^n(n-b) \sum\limits_{\substack{j=0\\ j\neq \langle -x\rangle_p}}^{p-1} (x+j)^b((x+j)^{p-1}-1)/p \pmod{p^{n+1}} & \\ \qquad\qquad \text{if } p-1 \mid n, & \\ p^n(n-b) \sum\limits_{\substack{j=0\\ j\neq \langle -x\rangle_p}}^{p-1} (x+j)^b \equiv -\chi(p-1 \mid b)(n-b)p^n \pmod{p^{n+1}} & \\ \qquad\qquad \text{if } p-1 \mid n+1, & \\ 0 \pmod{p^{n+1}} \qquad \text{if } p-1 \nmid n \text{ and } p-1 \nmid n+1. & \end{cases}\end{aligned}$$

On the other hand, for $t \in \mathbb{Z}_p$ we have $B_n(t) - B_n \in \mathbb{Z}_p$ (cf. [S4, Lemma 2.3]) and so

$$(-np)p^n B_n(x_j) \equiv -np^{n+1}B_n \equiv \begin{cases} np^n \pmod{p^{n+1}} & \text{if } p-1 \mid n,\\ 0 \pmod{p^{n+1}} & \text{if } p-1 \nmid n.\end{cases}$$

Thus applying (6.2) we get

$$\sum_{\substack{j=0\\ j\ne\langle -x\rangle_p}}^{p-1} (x+j)^{b-n}\cdot(-np)p^nB_n(x_j)$$

$$\equiv\begin{cases}\sum\limits_{\substack{j=0\\ j\ne\langle -x\rangle_p}}^{p-1} (x+j)^b\cdot np^n\equiv -np^n\chi(p-1\mid b)\pmod{p^{n+1}} & \text{if } p-1\mid n,\\ 0\pmod{p^{n+1}} & \text{if } p-1\nmid n.\end{cases}$$

Hence, by the above and the fact $(1-p)^n\equiv 1-np \pmod{p^2}$ we obtain

$$\sum_{\substack{j=0\\ j\ne\langle -x\rangle_p}}^{p-1} (x+j)^{b-n}(1-p)^np^nB_n(x_j)-\sum_{\substack{j=0\\ j\ne\langle -x\rangle_p}}^{p-1} (x+j)^{b-n}p^nB_n(x_j)$$

$$\equiv \sum_{\substack{j=0\\ j\ne\langle -x\rangle_p}}^{p-1} (x+j)^{b-n}\cdot(-np)p^nB_n(x_j)$$

$$\equiv\begin{cases}-np^n\pmod{p^{n+1}} & \text{if } p-1\mid b \text{ and } p-1\mid n,\\ 0\pmod{p^{n+1}} & \text{if } p-1\nmid b \text{ or } p-1\nmid n.\end{cases}$$

Now combining the above we see that

$$S_n-\sum_{\substack{j=0\\ j\ne\langle -x\rangle_p}}^{p-1} (x+j)^{b-n}p^nB_n(x_j)$$

$$\equiv\begin{cases}-np^n+(n-b)p^nT\pmod{p^{n+1}} & \text{if } p-1\mid b \text{ and } p-1\mid n,\\ p^n(n-b)T\pmod{p^{n+1}} & \text{if } p-1\nmid b \text{ and } p-1\mid n,\\ p^n(b-n)\pmod{p^{n+1}} & \text{if } p-1\mid b \text{ and } p-1\mid n+1,\\ 0\pmod{p^{n+1}} & \text{otherwise.}\end{cases}$$

This is the result.

Remark 6.1 When $p=2$, $b\ge 1$ and $n\ge 2$, setting $\Delta(b,n,p)=b-n$ we can show that the result of Proposition 6.1 is also true.

Theorem 6.1. *Let p be a prime greater than 3, $x\in\mathbb{Z}_p$, $n\in\mathbb{N}$, $n\not\equiv 0,1 \pmod{p-1}$ and $b\in\{0,1,2,\dots\}$. Let n_0 be given by $n\equiv n_0 \pmod{p-1}$ and $n_0\in\{2,3,\dots,p-2\}$. Set*

$$S_n=\sum_{k=0}^n\binom{n}{k}(-1)^k\Big(pB_{k(p-1)+b}(x)-p^{k(p-1)+b}B_{k(p-1)+b}\Big(\frac{x+\langle -x\rangle_p}{p}\Big)\Big).$$

Then

$$S_n \equiv \begin{cases} \left(\frac{n}{n_0}\cdot\frac{S_{n_0}}{p^{n_0}}+\frac{(n+2)b}{2}\right)p^n \pmod{p^{n+1}} & \text{if } p-1\mid b \text{ and } p-1\mid n+1,\\ \frac{n}{n_0}\cdot\frac{S_{n_0}}{p^{n_0}}\cdot p^n \pmod{p^{n+1}} & \text{if } p-1\nmid b \text{ or } p-1\nmid n+1.\end{cases}$$

Proof. Since $p-1\nmid n$ we know that $B_n/n\in\mathbb{Z}_p$. For $t\in\mathbb{Z}_p$, by [S4, Lemma 2.3] we have $(B_n(t)-B_n)/n\in\mathbb{Z}_p$. Thus

$$\frac{B_n(t)}{n}=\frac{B_n(t)-B_n}{n}+\frac{B_n}{n}\in\mathbb{Z}_p.$$

As $n\not\equiv 0,1 \pmod{p-1}$, by [S5, Corollary 3.1] we have

$$\frac{B_n(t)}{n}\equiv\frac{B_{n_0}(t)-p^{n_0-1}B_{n_0}\big((t+\langle -t\rangle_p)/p\big)}{n_0}\equiv\frac{B_{n_0}(t)}{n_0} \pmod p.$$

Set $x_j=((x+j)^p-(x+j))/(p(p-1))$. Then $x_j\in\mathbb{Z}_p$. Thus $B_n(x_j)/n\in\mathbb{Z}_p$ and $B_n(x_j)/n\equiv B_{n_0}(x_j)/n_0 \pmod p$. From Proposition 6.1 and the above we see that

$$\begin{aligned}\frac{S_n}{p^n}&\equiv\sum_{\substack{j=0\\ j\ne\langle -x\rangle_p}}^{p-1}(x+j)^{b-n}B_n(x_j)+(b-n)\chi(p-1\mid b)\chi(p-1\mid n+1)\\ &\equiv n\sum_{\substack{j=0\\ j\ne\langle -x\rangle_p}}^{p-1}(x+j)^{b-n_0}\frac{B_{n_0}(x_j)}{n_0}+(b-n)\chi(p-1\mid b)\chi(p-1\mid n+1) \pmod p\end{aligned}$$

and so

$$\frac{S_{n_0}}{p^{n_0}}\equiv n_0\sum_{\substack{j=0\\ j\ne\langle -x\rangle_p}}^{p-1}(x+j)^{b-n_0}\frac{B_{n_0}(x_j)}{n_0}+(b-n_0)\chi(p-1\mid b)\chi(p-1\mid n+1) \pmod p.$$

Thus

$$\begin{aligned}\frac{S_n}{p^n}&\equiv\frac{n}{n_0}\Big(\frac{S_{n_0}}{p^{n_0}}-(b-n_0)\chi(p-1\mid b)\chi(p-1\mid n+1)\Big)\\ &\qquad+(b-n)\chi(p-1\mid b)\chi(p-1\mid n+1)\\ &=\frac{n}{n_0}\cdot\frac{S_{n_0}}{p^{n_0}}+b\Big(1-\frac{n}{n_0}\Big)\chi(p-1\mid b)\chi(p-1\mid n+1)\\ &\equiv\frac{n}{n_0}\cdot\frac{S_{n_0}}{p^{n_0}}+b\Big(1+\frac{n}{2}\Big)\chi(p-1\mid b)\chi(p-1\mid n+1) \pmod p.\end{aligned}$$

This proves the theorem.

Theorem 6.2. *Let p be an odd prime, $x \in \mathbb{Z}_p$, $b, n \in \mathbb{Z}$ with $n \geq 1$ and $b \geq 0$. If $p \mid n$ and $p-1 \nmid n$, then*

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \Big(pB_{k(p-1)+b}(x) - p^{k(p-1)+b} B_{k(p-1)+b}\Big(\frac{x + \langle -x \rangle_p}{p} \Big) \Big)$$
$$\equiv \begin{cases} bp^n \pmod{p^{n+1}} & \text{if } p-1 \mid b \text{ and } p-1 \mid n+1, \\ 0 \pmod{p^{n+1}} & \text{if } p-1 \nmid b \text{ or } p-1 \nmid n+1. \end{cases}$$

Proof. As $p-1 \nmid n$ and $p \mid n$, for $t \in \mathbb{Z}_p$ we see that $B_n(t)/n \in \mathbb{Z}_p$ and so $B_n(t) = nB_n(t)/n \equiv 0 \pmod p$. Thus the result follows from Proposition 6.1.

Theorem 6.3. *Let p be an odd prime, $n \in \mathbb{N}$ and $b \in \{0, 2, 4, \dots\}$. If $p(p-1) \mid n$, then*

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k (1 - p^{k(p-1)+b-1}) pB_{k(p-1)+b}$$
$$\equiv \begin{cases} p^{n-1} - 2p^n \pmod{p^{n+1}} & \text{if } p-1 \mid b, \\ 0 \pmod{p^{n+1}} & \text{if } p-1 \nmid b. \end{cases}$$

Proof. From Proposition 6.1 we see that

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k (1 - p^{k(p-1)+b-1}) pB_{k(p-1)+b}$$
$$\equiv \sum_{j=1}^{p-1} j^{b-n} p^n B_n\Big(\frac{j^p - j}{p(p-1)} \Big) - bTp^n \pmod{p^{n+1}},$$

where

$$T = \sum_{j=1}^{p-1} \frac{j^{p-1+b} - j^b}{p}.$$

For $p > 3$ and $m \in \mathbb{N}$, from [S5, (5.1)] we have

$$\sum_{j=1}^{p-1} j^m \equiv pB_m + \frac{p^2}{2} mB_{m-1} + \frac{p^3}{6} m(m-1) B_{m-2} \pmod{p^3}.$$

If $m \geq 4$ is even, then $B_{m-1} = 0$ and $pB_{m-2} \in \mathbb{Z}_p$. Thus

$$\sum_{j=1}^{p-1} j^m \equiv pB_m \pmod{p^2} \quad \text{for} \quad m = 2, 4, 6, \dots \tag{6.3}$$

Hence

$$T \equiv \begin{cases} \frac{pB_{p-1+b} - pB_b}{p} \pmod p & \text{if } p > 3 \text{ and } b > 0, \\ \frac{pB_{p-1} - (p-1)}{p} \pmod p & \text{if } p > 3 \text{ and } b = 0, \\ \frac{2^{2+b} - 2^b}{3} = 2^b \equiv (-1)^b = 1 \equiv \frac{3B_2 - 2}{3} \pmod 3 & \text{if } p = 3. \end{cases}$$

If $p>3$ and $b=k(p-1)$ for some $k\in\mathbb{N}$, by [S4, Corollary 4.2] we have

$$pB_b=pB_{k(p-1)}\equiv kpB_{p-1}-(k-1)(p-1) \pmod{p^2} \tag{6.4}$$

and

$$pB_{p-1+b}=pB_{(k+1)(p-1)}\equiv (k+1)pB_{p-1}-k(p-1) \pmod{p^2}.$$

Thus

$$T\equiv \frac{pB_{p-1+b}-pB_b}{p}\equiv \frac{pB_{p-1}-(p-1)}{p} \pmod p.$$

If $p>3$ and $p-1\nmid b$, by Kummer's congruences we have

$$\frac{B_{p-1+b}}{p-1+b}\equiv \frac{B_b}{b} \pmod p \quad \text{and so} \quad B_{p-1+b}\equiv (b-1)\frac{B_b}{b} \pmod p.$$

Thus

$$T\equiv \frac{pB_{p-1+b}-pB_b}{p}\equiv \frac{b-1}{b}B_b-B_b=-\frac{B_b}{b} \pmod p.$$

Summarizing the above we have

$$T\equiv \begin{cases} \frac{pB_{p-1}-(p-1)}{p} \pmod p & \text{if } p-1\mid b,\\ -\frac{B_b}{b} \pmod p & \text{if } p-1\nmid b. \end{cases} \tag{6.5}$$

As $p(p-1)\mid n$, from Corollary 5.1 we have $pB_n(x)\equiv p-1 \pmod{p^2}$ for $x\in\mathbb{Z}_p$. Note that $j^n\equiv 1 \pmod{p^2}$ for $j=1,2,\ldots,p-1$. Combining the above we obtain

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}(-1)^k(1-p^{k(p-1)+b-1})pB_{k(p-1)+b}\\ &\equiv \sum_{j=1}^{p-1} j^{b-n}p^{n-1}\cdot pB_n\Big(\frac{j^p-j}{p(p-1)}\Big)-bTp^n\\ &\equiv \sum_{j=1}^{p-1} j^b p^{n-1}(p-1)-bTp^n \pmod{p^{n+1}}.\end{aligned}$$

From (6.3) and (6.4) we see that

$$\sum_{j=1}^{p-1} j^b\equiv \begin{cases} pB_b\equiv \frac{b}{p-1}\cdot pB_{p-1}-(\frac{b}{p-1}-1)(p-1) \pmod{p^2} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } p>3,\ b>0 \text{ and } p-1\mid b,\\ pB_b \pmod{p^2} \qquad\qquad\qquad \text{if } p>3 \text{ and } p-1\nmid b,\\ p-1 \pmod{p^2} \qquad\qquad\qquad \text{if } p>3 \text{ and } b=0,\\ 1+(1+3)^{\frac{b}{2}}\equiv 2+\frac{3b}{2}\equiv 2+6b \pmod 9 \quad \text{if } p=3. \end{cases}$$

That is,

$$\sum_{j=1}^{p-1} j^b \equiv \begin{cases} \frac{b}{p-1}(pB_{p-1}-(p-1))+p-1 \pmod{p^2} & \text{if } p-1 \mid b, \\ pB_b \pmod{p^2} & \text{if } p-1 \nmid b. \end{cases}$$

Hence

$$\begin{aligned} &\sum_{k=0}^{n}\binom{n}{k}(-1)^k(1-p^{k(p-1)+b-1})pB_{k(p-1)+b} \\ &\equiv p^{n-1}(p-1)\sum_{j=1}^{p-1} j^b - bTp^n \\ &\equiv \begin{cases} p^{n-1}(b(pB_{p-1}-(p-1))+(p-1)^2)-p^{n-1}b(pB_{p-1}-(p-1)) \\ \qquad = p^{n-1}(p-1)^2 \equiv p^{n-1}-2p^n \pmod{p^{n+1}} & \text{if } p-1 \mid b, \\ p^{n-1}(p-1)\cdot pB_b - bp^n\cdot(-\frac{B_b}{b}) = p^{n+1}B_b \equiv 0 \pmod{p^{n+1}} & \text{if } p-1 \nmid b. \end{cases} \end{aligned}$$

This completes the proof.

Theorem 6.4. *Let p be a prime greater than 3, $x \in \mathbb{Z}_p$, $n \in \mathbb{N}$, $n \not\equiv 0,1 \pmod{p-1}$ and $b \in \{0,1,2,\dots\}$. Let n_0 be given by $n \equiv n_0 \pmod{p-1}$ and $n_0 \in \{2,3,\dots,p-2\}$. Let*

$$f(k) = pB_{k(p-1)+b}(x) - p^{k(p-1)+b}B_{k(p-1)+b}\Big(\frac{x+\langle -x\rangle_p}{p}\Big).$$

Then for $k=0,1,2,\dots$ we have

$$\begin{aligned} f(k) \equiv &\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}f(r) + \frac{n}{n_0}\cdot\frac{\sum_{s=0}^{n_0}\binom{n_0}{s}(-1)^s f(s)}{p^{n_0}}\binom{k}{n}(-p)^n \\ &+ \chi(p-1 \mid n+1)\chi(p-1 \mid b)\left(\frac{(n+2)b}{2}\binom{k}{n} - \binom{k}{n+1}\right)(-p)^n \pmod{p^{n+1}}. \end{aligned}$$

Proof. From [S4, Theorem 3.1] we have

$$\sum_{k=0}^{m}\binom{m}{k}(-1)^k f(k) \equiv p^{m-1}\chi(p-1 \mid m)\chi(p-1 \mid b) \pmod{p^m} \quad \text{for} \quad m \in \mathbb{N}.$$

Thus applying [S4, Lemma 2.1], Theorem 6.1 and the above we see that

$$\begin{aligned}
&f(k)-\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}f(r)\\
&=\sum_{r=n}^{k}\binom{k}{r}(-1)^r\sum_{s=0}^{r}\binom{r}{s}(-1)^s f(s)\\
&\equiv\binom{k}{n}(-1)^n\sum_{s=0}^{n}\binom{n}{s}(-1)^s f(s)+\binom{k}{n+1}(-1)^{n+1}\sum_{s=0}^{n+1}\binom{n+1}{s}(-1)^s f(s)\\
&\equiv\binom{k}{n}(-1)^n p^n\Big(\frac{n}{n_0}\cdot\frac{\sum_{s=0}^{n_0}\binom{n_0}{s}(-1)^s f(s)}{p^{n_0}}+\frac{(n+2)b}{2}\chi(p-1\mid n+1)\chi(p-1\mid b)\Big)\\
&\quad+\binom{k}{n+1}(-1)^{n+1}p^n\chi(p-1\mid n+1)\chi(p-1\mid b)\pmod{p^{n+1}}.
\end{aligned}$$

This yields the result.

Corollary 6.1. *Let $k,n\in\mathbb{N}$.*

(i) *If $n\equiv 2\pmod 4$, then*

$$\begin{aligned}(5-5^{4k})B_{4k}\equiv&\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}(5-5^{4r})B_{4r}\\&+3n\binom{k}{n}5^n\pmod{5^{n+1}}\end{aligned}$$

and

$$\begin{aligned}(5-5^{4k+2})B_{4k+2}\equiv&\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}(5-5^{4r+2})B_{4r+2}\\&-n\binom{k}{n}5^n\pmod{5^{n+1}}.\end{aligned}$$

(ii) *If $n\equiv 3\pmod 4$, then*

$$\begin{aligned}(5-5^{4k})B_{4k}\equiv&\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}(5-5^{4r})B_{4r}\\&+\binom{k}{n+1}5^n\pmod{5^{n+1}}\end{aligned}$$

and

$$\begin{aligned}(5-5^{4k+2})B_{4k+2}\equiv&\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}(5-5^{4r+2})B_{4r+2}\\&+n\binom{k}{n}5^n\pmod{5^{n+1}}.\end{aligned}$$

7. Congruences for Euler numbers.

We recall that the Euler numbers $\{E_n\}$ are given by

$$E_0=1,\ E_{2n-1}=0 \quad \text{and} \quad \sum_{r=0}^{n}\binom{2n}{2r}E_{2r}=0\ (n\ge 1).$$

The first few Euler numbers are shown below:

$$E_0=1,\ E_2=-1,\ E_4=5,\ E_6=-61,\ E_8=1385,\ E_{10}=-50521,$$
$$E_{12}=2702765,\ E_{14}=-199360981,\ E_{16}=19391512145.$$

By (1.2) and (2.9) we have

$$\begin{aligned} E_{2n}&=2^{2n}E_{2n}\Big(\frac12\Big)=2^{2n}\cdot\frac{2^{2n+1}}{2n+1}\Big(B_{2n+1}\Big(\frac34\Big)-B_{2n+1}\Big(\frac14\Big)\Big)\\ &=\frac{2^{4n+1}}{2n+1}\Big(-B_{2n+1}\Big(\frac14\Big)-B_{2n+1}\Big(\frac14\Big)\Big). \end{aligned}$$

That is,

$$E_{2n}=-4^{2n+1}\frac{B_{2n+1}(\frac14)}{2n+1}. \tag{7.1}$$

Lemma 7.1. *Let p be an odd prime and $b\in\{0,2,4,\ldots\}$. Then $f(k)=(1-(-1)^{\frac{p-1}{2}}p^{k(p-1)+b})E_{k(p-1)+b}$ is a $p-$regular function.*

Proof. As $p>2$ and $2\mid b$ we see that $p-1\nmid b+1$. For $x\in\mathbb{Z}_p$, from Lemma 5.1(i) we know that $F(k)=(B_{k(p-1)+b+1}(x)-p^{k(p-1)+b}B_{k(p-1)+b+1}(x'))/(k(p-1)+b+1)$ is a $p-$regular function, where $x'=(x+\langle -x\rangle_p)/p$. It is clear that

$$\frac{\frac14+\langle-\frac14\rangle_p}{p}=\begin{cases}\frac1p(\frac14+\frac{p-1}{4})=\frac14 & \text{if } p\equiv 1 \pmod 4,\\ \frac1p(\frac14+\frac{3p-1}{4})=\frac34 & \text{if } p\equiv 3 \pmod 4).\end{cases}$$

Thus, using (2.9) we see that

$$B_{k(p-1)+b+1}\Big(\frac{\frac14+\langle-\frac14\rangle_p}{p}\Big)=B_{k(p-1)+b+1}\Big(\Big\{\frac p4\Big\}\Big)=(-1)^{\frac{p-1}{2}}B_{k(p-1)+b+1}\Big(\frac14\Big).$$

Hence

$$\begin{aligned} g(k)&=\big(1-(-1)^{\frac{p-1}{2}}p^{k(p-1)+b}\big)\frac{B_{k(p-1)+b+1}(\frac14)}{k(p-1)+b+1}\\ &=-4^{-(k(p-1)+b+1)}\big(1-(-1)^{\frac{p-1}{2}}p^{k(p-1)+b}\big)E_{k(p-1)+b} \end{aligned}$$

is a $p-$regular function. For $n\in\mathbb{N}$ we see that

$$\sum_{k=0}^{n}\binom nk(-1)^k\big(-4^{k(p-1)+b+1}\big)=-4^{b+1}(1-4^{p-1})^n\equiv 0 \pmod{p^n}.$$

Namely, $-4^{k(p-1)+b+1}$ is a $p-$regular function. Hence, using [S5, Theorem 2.3] we see that $f(k)=-4^{k(p-1)+b+1}g(k)$ is also a $p-$regular function. This proves the lemma.

From Lemma 7.1 and Theorem 4.3 we have:

Theorem 7.1. *Let p be an odd prime, $k,m,n,t\in\mathbb{N}$ and $b\in\{0,2,4,\dots\}$. Then*

$$\begin{aligned}&\big(1-(-1)^{\frac{p-1}{2}}p^{ktp^{m-1}(p-1)+b}\big)E_{ktp^{m-1}(p-1)+b}\\ &\equiv\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}\big(1-(-1)^{\frac{p-1}{2}}p^{rtp^{m-1}(p-1)+b}\big)\\ &\qquad\times E_{rtp^{m-1}(p-1)+b}\pmod{p^{mn}}.\end{aligned}$$

Putting $n=1,2,3$ and $t=1$ in Theorem 7.1 we obtain the following result.

Corollary 7.1. *Let p be an odd prime, $k,m\in\mathbb{N}$ and $b\in\{0,2,4,\dots\}$. Then*
(i) ([C, p. 131]) $E_{k\varphi(p^m)+b}\equiv\big(1-(-1)^{\frac{p-1}{2}}p^b\big)E_b\pmod{p^m}$.
(ii) $E_{k\varphi(p^m)+b}\equiv kE_{\varphi(p^m)+b}-(k-1)\big(1-(-1)^{\frac{p-1}{2}}p^b\big)E_b\pmod{p^{2m}}$.
(iii) *We have*

$$\begin{aligned}E_{k\varphi(p^m)+b}\equiv{}&\frac{k(k-1)}{2}E_{2\varphi(p^m)+b}-k(k-2)\big(1-(-1)^{\frac{p-1}{2}}p^{\varphi(p^m)+b}\big)E_{\varphi(p^m)+b}\\ &+\frac{(k-1)(k-2)}{2}\big(1-(-1)^{\frac{p-1}{2}}p^b\big)E_b\pmod{p^{3m}}.\end{aligned}$$

From Lemma 7.1 and Corollary 4.2(iv) we have:

Theorem 7.2. *Let p be an odd prime, $k,m\in\mathbb{N}$ and $b\in\{0,2,4,\dots\}$. Then*

$$E_{k\varphi(p^m)+b}\equiv(1-kp^{m-1})(1-(-1)^{\frac{p-1}{2}}p^b)E_b+kp^{m-1}E_{p-1+b}\pmod{p^{m+1}}.$$

Corollary 7.2. *Let p be an odd prime and $k,m\in\mathbb{N}$. Then*

$$E_{k\varphi(p^m)}\equiv\begin{cases}kp^{m-1}E_{p-1}\pmod{p^{m+1}} & \text{if } p\equiv1\pmod 4,\\ 2+kp^{m-1}(E_{p-1}-2)\pmod{p^{m+1}} & \text{if } p\equiv3\pmod 4.\end{cases}$$

From [S5, Theorem 2.1] and Lemma 7.1 we have:

Theorem 7.3. *Let p be an odd prime, $n\in\mathbb{N}$ and $b\in\{0,2,4,\dots\}$. Then there are integers $a_0,a_1,\dots,a_{n-1}$ such that*

$$(1-(-1)^{\frac{p-1}{2}}p^{k(p-1)+b})E_{k(p-1)+b}\equiv a_{n-1}k^{n-1}+\cdots+a_1k+a_0\pmod{p^n}$$

for every $k=0,1,2,\dots$ Moreover, if $p\ge n$, then $a_0,a_1,\dots,a_{n-1}\pmod{p^n}$ are uniquely determined.

As examples, we have

$$(1+3^{2k})E_{2k}\equiv-12k+2\pmod{3^3},\tag{7.2}$$

$$(1-5^{4k})E_{4k}\equiv-750k^3+1375k^2-620k\pmod{5^5},\tag{7.3}$$

$$(1-5^{4k+2})E_{4k+2}\equiv1000k^3+1500k^2+540k+24\pmod{5^5}.\tag{7.4}$$

Theorem 7.4. *Let $n \in \mathbb{N}$ and $b \in \{0, 2, 4, \dots\}$. Suppose $\alpha_n \in \mathbb{N}$ and $2^{\alpha_n - 1} \le n < 2^{\alpha_n}$. Then*

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k E_{2k+b} \equiv 0 \pmod{2^{2n-\alpha_n}}.$$

Proof. We first prove the result in the case $b = 0$. Taking $x = 0$ in (1.2) we find

$$\sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} E_r = \frac{2^{n+1}}{n+1} \big(B_{n+1} - 2^{n+1} B_{n+1}\big).$$

Thus applying the binomial inversion formula we have

$$E_n = \sum_{m=0}^{n} \binom{n}{m} \frac{2^{m+1}(1 - 2^{m+1})}{m+1} B_{m+1}.$$

Using this we see that

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} E_{2k} &= \sum_{k=0}^{n} \sum_{m=0}^{2k} \binom{n}{k} (-1)^{n-k} \binom{2k}{m} \frac{2^{m+1}(1 - 2^{m+1})}{m+1} B_{m+1} \\
&= \sum_{m=0}^{2n} \frac{2^{m+1}(1 - 2^{m+1})}{m+1} B_{m+1} \sum_{\frac{m}{2} \le k \le n} \binom{n}{k} (-1)^{n-k} \binom{2k}{m} \\
&= \sum_{m=1}^{2n} \frac{2^{m+1}(1 - 2^{m+1})}{m+1} B_{m+1} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \binom{2k}{m}.
\end{aligned}$$

By Lemma 4.1 we have

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \binom{2k}{m} &= \frac{n!}{m!} \sum_{j=n}^{m} (-1)^{m-j} s(m, j) S(j, n) \cdot 2^j \\
&= \sum_{j=n}^{m} (-1)^{m-j} \frac{j! s(m, j)}{m!} 2^{m-j} \cdot \frac{n! S(j, n)}{j!} 2^{j-n} \cdot 2^{j+n-m}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} E_{2k} \\
&= \sum_{m=1}^{2n} \frac{2^{m+1}(1 - 2^{m+1})}{m+1} B_{m+1} \sum_{j=n}^{m} (-1)^{m-j} \frac{j! s(m, j)}{m!} 2^{m-j} \cdot \frac{n! S(j, n)}{j!} 2^{j-n} \cdot 2^{j+n-m} \\
&= \sum_{m=n}^{2n} \frac{2^{m+1}(1 - 2^{m+1})}{m+1} B_{m+1} \sum_{j=n}^{m} (-1)^{m-j} \frac{j! s(m, j)}{m!} 2^{m-j} \cdot \frac{n! S(j, n)}{j!} 2^{j-n} \cdot 2^{j+n-m}.
\end{aligned}$$

It is well known that $2B_k \in \mathbb{Z}_2$. Suppose $2^{\mathrm{ord}_2(m+1)} \parallel m+1$. We then have

$$\frac{1}{2^{m-\mathrm{ord}_2(m+1)}} \cdot \frac{2^{m+1}B_{m+1}}{m+1} = \frac{2B_{m+1}}{2^{-\mathrm{ord}_2(m+1)}(m+1)} \in \mathbb{Z}_2.$$

On the other hand, by Lemma 4.2 we have $\frac{j!s(m,j)}{m!}2^{m-j} \in \mathbb{Z}_2$ and $\frac{n!S(j,n)}{j!}2^{j-n} \in \mathbb{Z}_2$. Hence, if $n \le j \le m \le 2n$, then

$$\begin{aligned}&\frac{2^{m+1}(1-2^{m+1})}{m+1}B_{m+1} \cdot (-1)^{m-j}\frac{j!s(m,j)}{m!}2^{m-j} \cdot \frac{n!S(j,n)}{j!}2^{j-n} \cdot 2^{j+n-m}\\ &\equiv 0 \pmod{2^{j+n-\mathrm{ord}_2(m+1)}}.\end{aligned}$$

When $n \le j \le m \le 2n$, we also have $m+1 < 2(n+1) \le 2^{\alpha_n+1}$ and so $\mathrm{ord}_2(m+1) \le \alpha_n$, thus $j+n-\mathrm{ord}_2(m+1) \ge j+n-\alpha_n \ge 2n-\alpha_n$. Therefore, by the above we obtain $\sum_{k=0}^{n}\binom{n}{k}(-1)^k E_{2k} \equiv 0 \pmod{2^{2n-\alpha_n}}$. So the result holds for $b=0$.

From [S5, (2.5)] we know that for any function f,

$$(7.5)\qquad \sum_{k=0}^{n}\binom{n}{k}(-1)^k f(k+m) = \sum_{k=0}^{m}\binom{m}{k}(-1)^k \sum_{r=0}^{k+n}\binom{k+n}{r}(-1)^r f(r).$$

Thus,

$$(7.6)\qquad \sum_{k=0}^{n}\binom{n}{k}(-1)^k E_{2k+b} = \sum_{k=0}^{b/2}\binom{\frac{b}{2}}{k}(-1)^k \sum_{r=0}^{k+n}\binom{k+n}{r}(-1)^r E_{2r}.$$

As $\alpha_{s+1} = \alpha_s$ or α_s+1, we see that $2(s+1)-\alpha_{s+1} \ge 2s-\alpha_s$ and hence $2r-\alpha_r \ge 2s-\alpha_s$ for $r \ge s$. As the result holds for $b=0$ we have

$$\sum_{r=0}^{k+n}\binom{k+n}{r}(-1)^r E_{2r} \equiv 0 \pmod{2^{2(k+n)-\alpha_{k+n}}}.$$

Since $2(k+n)-\alpha_{k+n} \ge 2n-\alpha_n$, we must have $\sum_{r=0}^{k+n}\binom{k+n}{r}(-1)^r E_{2r} \equiv 0 \pmod{2^{2n-\alpha_n}}$. Hence applying (7.6) we obtain

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k E_{2k+b} \equiv 0 \pmod{2^{2n-\alpha_n}}.$$

This proves the theorem.

Corollary 7.3. *Let $n \in \mathbb{N}$ and $b \in \{0, 2, 4, \dots\}$. Then*

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k E_{2k+b} \equiv \begin{cases} 2 \pmod 4 & \text{if } n = 1, \\ 0 \pmod{2^{n+1}} & \text{if } n > 1 \end{cases}$$

and thus $f(k) = E_{2k+b}$ is a 2−regular function.

Proof. Suppose $\alpha_n \in \mathbb{N}$ and $2^{\alpha_n - 1} \le n < 2^{\alpha_n}$. By Theorem 7.4 we have

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k E_{2k+b} \equiv 0 \pmod{2^{2n-\alpha_n}}.$$

If $\alpha_n \ge n$, then $2^{n-1} \le 2^{\alpha_n - 1} \le n$. For $n \ge 3$ we have $2^{n-1} > n$, thus $\alpha_n < n$ and hence $2n - \alpha_n \ge n + 1$. Therefore, for $n \ge 3$ we have $\sum_{k=0}^{n} \binom{n}{k} (-1)^k E_{2k+b} \equiv 0 \pmod{2^{n+1}}$. As $E_0 - E_2 = 1 - (-1) = 2$ and $E_0 - 2E_2 + E_4 = 1 - 2(-1) + 5 = 8$, applying (7.6) and the above we see that $E_b - E_{b+2} \equiv E_0 - E_2 = 2 \pmod 8$ and $E_b - 2E_{b+2} + E_{b+4} \equiv 0 \pmod 8$. So the result follows.

Theorem 7.5. *Suppose $k, m, n, t \in \mathbb{N}$ and $b \in \{0, 2, 4, \dots\}$. For $s \in \mathbb{N}$ let $\alpha_s \in \mathbb{N}$ be given by $2^{\alpha_s - 1} \le s < 2^{\alpha_s}$ and let $e_s = 2^{-s} \sum_{r=0}^{s} \binom{s}{r} (-1)^r E_{2r}$. Then*

$$\begin{aligned} E_{2^m kt+b} \equiv & \sum_{r=0}^{n-1} (-1)^{n-1-r} \binom{k-1-r}{n-1-r} \binom{k}{r} E_{2^m rt+b} \\ & + 2^{mn} \binom{k}{n} (-t)^n e_n \pmod{2^{mn+n+1-\alpha_{n+1}}}. \end{aligned}$$

Moreover, for $m \ge 2$ we have

$$\begin{aligned} E_{2^m kt+b} \equiv & \sum_{r=0}^{n-1} (-1)^{n-1-r} \binom{k-1-r}{n-1-r} \binom{k}{r} E_{2^m rt+b} \\ & + 2^{mn} \binom{k}{n} (-t)^n \Big(e_n + n e_{n+1} + \frac{n(n-1)}{2} e_{n+2} \Big) \pmod{2^{mn+n+2-\alpha_{n+1}}}. \end{aligned}$$

Proof. For $s \in \mathbb{N}$ set $A_s = 2^{-s} \sum_{r=0}^{s} \binom{s}{r} (-1)^r E_{2r+b}$. Since $\alpha_s \le s$, by Theorem 7.4 we have $A_s \in \mathbb{Z}_2$ and $2^{s-\alpha_s} \mid A_s$. As $\alpha_{s+1} \le \alpha_s + 1$ we have $s + 1 - \alpha_{s+1} \ge s - \alpha_s$ and hence $r - \alpha_r \ge s - \alpha_s$ for $r \ge s$. Therefore $2^s - \alpha_s \mid A_r$ for $r \ge s$. As $1 + \alpha_{n+1} \ge \alpha_{n+3}$ we see that $n + 3 - \alpha_{n+3} \ge n + 2 - \alpha_{n+1}$ and thus $2^{n+2-\alpha_{n+1}} \mid A_r$ for $r \ge n + 3$. By (7.6) we have

$$A_n = \sum_{k=0}^{b/2} \binom{\frac{b}{2}}{k} (-1)^k 2^k e_{k+n}.$$

Since $2^{n+2-\alpha_{n+1}} \mid e_r$ for $r \ge n+3$, $2^{n+2-\alpha_{n+1}} \mid 2e_{n+1}$ and $2^{n+2-\alpha_{n+1}} \mid 2^2 e_{n+2}$, we see that $A_n \equiv e_n \pmod{2^{n+2-\alpha_{n+1}}}$.

From Corollary 7.3 and the proof of Theorem 4.2 we know that

$$\begin{aligned}&\sum_{r=0}^{n}\binom{n}{r}(-1)^r E_{2\cdot 2^{m-1}rt+b}\\&=A_n t^n\cdot 2^{mn}+\sum_{r=n+1}^{2^{m-1}nt}(-2)^n(-1)^r A_r\Big(\frac{(-1)^{r-n}s(r,n)n!}{r!}2^{r-n}\cdot 2^{(m-1)n}t^n\\&\quad+\sum_{j=n+1}^{r}\frac{(-1)^{r-j}s(r,j)j!}{r!}2^{r-j}\cdot\frac{S(j,n)n!}{j!}2^{j-n}\cdot(2^{m-1}t)^j\Big).\end{aligned}$$

By Lemma 4.2, for $n+1\le j\le r$ we have

$$\frac{s(r,j)j!}{r!}2^{r-j},\ \frac{S(j,n)n!}{j!}2^{j-n}\in\mathbb{Z}_2\quad\text{and}\quad\frac{s(r,n)n!}{r!}2^{r-n}\equiv\binom{n}{r-n}\pmod 2.$$

As $2^{n+1-\alpha_{n+1}}\mid A_r$ for $r\ge n+1$, by the above we obtain

$$\sum_{r=0}^{n}\binom{n}{r}(-1)^r E_{2^m rt+b}\equiv 2^{mn}A_n t^n\equiv 2^{mn}t^n e_n \pmod{2^{mn+n+1-\alpha_{n+1}}} \tag{7.7}$$

and so

$$\sum_{r=0}^{n}\binom{n}{r}(-1)^r E_{2^m rt+b}\equiv 0 \pmod{2^{mn+n-\alpha_n}}. \tag{7.8}$$

For $r\ge n+1$ we have $mr+r-\alpha_r\ge m(n+1)+n+1-\alpha_{n+1}\ge mn+n+2-\alpha_{n+1}$. Thus, if $r\ge n+1$, by (7.8) we have

$$\sum_{s=0}^{r}\binom{r}{s}(-1)^s E_{2^m st+b}\equiv 0 \pmod{2^{mn+n+2-\alpha_{n+1}}}. \tag{7.9}$$

By (4.5) we have

$$\begin{aligned}E_{2^m kt+b}&=\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}E_{2^m rt+b}\\&\quad+\sum_{r=n}^{k}\binom{k}{r}(-1)^r\sum_{s=0}^{r}\binom{r}{s}(-1)^s E_{2^m st+b}.\end{aligned}$$

Hence, applying (7.9) we obtain

$$\begin{aligned}&E_{2^m kt+b}-\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}E_{2^m rt+b}\\&\equiv\binom{k}{n}(-1)^n\sum_{s=0}^{n}\binom{n}{s}(-1)^s E_{2^m st+b}\pmod{2^{mn+n+2-\alpha_{n+1}}}.\end{aligned} \tag{7.10}$$

In view of (7.7), we get

$$E_{2^m kt+b} \equiv \sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}E_{2^m rt+b} + \binom{k}{n}(-1)^n \cdot 2^{mn}t^n e_n \pmod{2^{mn+n+1-\alpha_{n+1}}}.$$

Now assume $m \geq 2$. Then $(m-1)(n+1)+n \geq mn+1$. From the above we see that

$$\begin{aligned}&\sum_{r=0}^{n}\binom{n}{r}(-1)^r E_{2^m rt+b}\\ &\equiv 2^{mn}A_n t^n + \sum_{r=n+1}^{2^{m-1}nt}(-2)^n(-1)^r A_r \cdot \frac{(-1)^{r-n}s(r,n)n!}{r!}2^{r-n}\cdot 2^{(m-1)n}t^n\\ &\equiv 2^{mn}t^n\Big(A_n + \sum_{r=n+1}^{2^{m-1}nt}\binom{n}{r-n}A_r\Big) \equiv 2^{mn}t^n \sum_{r=n}^{n+2}\binom{n}{r-n}A_r\\ &\equiv 2^{mn}t^n\Big(e_n + ne_{n+1} + \binom{n}{2}e_{n+2}\Big) \pmod{2^{mn+n+2-\alpha_{n+1}}}.\end{aligned}$$

This together with (7.10) yields the remaining result. Hence the proof is complete.

As $2^{n-\alpha_n} \mid e_n$ and $n+1-\alpha_{n+1} \geq n-\alpha_n$, by Theorem 7.5 we have:

Corollary 7.4. *Let $k,m,n,t \in \mathbb{N}$ and $b \in \{0,2,4,\dots\}$. Let $\alpha \in \mathbb{N}$ be given by $2^{\alpha-1} \leq n < 2^{\alpha}$. Then*

$$E_{2^m kt+b} \equiv \sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}E_{2^m rt+b} \pmod{2^{mn+n-\alpha}}.$$

Corollary 7.5. *Let $k,m \in \mathbb{N}$ and $b \in \{0,2,4,\dots\}$. Then*

$$E_{2^m k+b} \equiv 2^m k + E_b \pmod{2^{m+1}}.$$

Proof. Observe that $e_1 = 1$ and $e_2 = 2$. For $m \geq 2$, taking $n = t = 1$ in Theorem 7.5 we obtain

$$E_{2^m k+b} \equiv E_b + 2^m(-k)(e_1+e_2) \equiv 2^m k + E_b \pmod{2^{m+1}}.$$

So the result holds for $m \geq 2$. Now taking $m = 2$ and $b = 0, 2$ in the congruence we see that $E_{4k} \equiv 1+4k \pmod 8$ and $E_{4k+2} \equiv -1+4k \pmod 8$. Hence $E_{2k} \equiv (-1)^k \pmod 4$ and so $E_{2k+b} \equiv (-1)^{k+b/2} \equiv (-1)^{b/2} + 2k \equiv E_b + 2k \pmod 4$. So the result is also true for $m = 1$. This completes the proof.

Remark 7.1 Corollary 7.5 is equivalent to the following Stern's result (see [St]):

$$2^m \parallel E_{n_1} - E_{n_2} \iff 2^m \parallel n_1 - n_2.$$

Putting $n = 2$, $t = 1$ in Theorem 7.5 and noting that $e_2 = 2,\ e_3 = 10,\ e_4 = 104$ we obtain the following result.

Corollary 7.6. *Let $k, m \in \mathbb{N}$, $m \geq 2$ and $b \in \{0, 2, 4, \dots\}$. Then*

$$E_{2^m k+b} \equiv kE_{2^m+b} - (k-1)E_b + 2^{2m}k(k-1) \pmod{2^{2m+2}}.$$

Taking $m = 2$ and $b = 0, 2$ in Corollary 7.6 we get:

Corollary 7.7. *For $k \in \mathbb{N}$ we have*

$$E_{4k} \equiv \begin{cases} 4k+1 \pmod{64} & \text{if } k \equiv 0, 1 \pmod 4, \\ 4k+33 \pmod{64} & \text{if } k \equiv 2, 3 \pmod 4 \end{cases}$$

and

$$E_{4k+2} \equiv \begin{cases} 4k-1 \pmod{64} & \text{if } k \equiv 0, 1 \pmod 4, \\ 4k-33 \pmod{64} & \text{if } k \equiv 2, 3 \pmod 4. \end{cases}$$

Corollary 7.8. *Let $k, m \in \mathbb{N}$, $m \geq 2$ and $b \in \{0, 2, 4, \dots\}$. Let $\delta_k = 0$ or 1 according as $4 \nmid k-3$ or $4 \mid k-3$. Then*

$$E_{2^m k+b} \equiv \binom{k}{2} E_{2^{m+1}+b} - k(k-2)E_{2^m+b} + \binom{k-1}{2} E_b + 2^{3m+1}\delta_k \pmod{2^{3m+2}}.$$

Proof. Observe that $e_3 = 10$, $e_4 = 104$, $e_5 = 1816$ and $\binom{k}{3} \equiv \delta_k \pmod 2$. Taking $n = 3$ and $t = 1$ in Theorem 7.5 we obtain the result.

Taking $m = 2$, $b = 0, 2$ in Corollary 7.8 and noting that $E_8 \equiv 105 \pmod{256}$, $E_{10} \equiv -89 \pmod{256}$ we deduce:

Corollary 7.9. *Let $k \in \mathbb{N}$ and $\delta_k = 0$ or 1 according as $4 \nmid k-3$ or $4 \mid k-3$. Then*

$$E_{4k} \equiv 48k^2 - 44k + 1 + 128\delta_k \pmod{256} \text{ and } E_{4k+2} \equiv 16k^2 - 76k - 1 + 128\delta_k \pmod{256}.$$

Remark 7.2 Let $\{S_n\}$ be given by (3.1). From Remark 3.1 we know that $(-1)^k S_k$ is a 2-regular function and hence $f(k) = (-1)^{k+b} S_{k+b}$ is also a 2-regular function, where $b \in \{0, 1, 2, \dots\}$. Thus, by Corollary 4.2, for $m \geqslant 2$, $k \geqslant 1$ and $b \geqslant 0$ we have $S_{2^{m-1}k+b} \equiv S_b \pmod{2^m}$ and $S_{2^{m-1}k+b} \equiv S_b - 2^{m-2}k(S_{b+2} + 4S_{b+1} + 3S_b) \pmod{2^{m+1}}$.

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