p-regular functions and congruences for Bernoulli and Euler numbers

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Notation: $\mathbb{Z}$—the set of integers, $\mathbb{N}$—the set of positive integers, $[x]$— the greatest integer not exceeding x, $\{x\}$—the fractional part of x, $(\frac{a}{m})$—the Jacobi symbol, $\mathbb{Z}_p$—the set of rational $p-$adic integers.

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§ 1. Definition of $\{B_n\}, \{E_n\}$ and $\{U_n\}$

The Bernoulli numbers $B_0, B_1, B_2, \ldots$ are given by

$$B_0 = 1,\ \sum_{k=0}^{n-1} \binom{n}{k} B_k = 0\ (n \ge 2).$$

The first few Bernoulli numbers are given below:

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30},$$
$$B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30},\ B_{10} = \frac{5}{66},$$
$$B_{12} = -\frac{691}{2730}, \quad B_{14} = \frac{7}{6}, \quad B_{16} = -\frac{3617}{510}.$$

Basic properties of $\{B_n\}$:

$$B_{2n+1} = 0 \quad \text{for} \quad n \ge 1.$$

von Staudt-Clausen Theorem (1844):

$$B_{2n} + \sum_{p-1|2n} \frac{1}{p} \in \mathbb{Z},$$

where p runs over all distinct primes satisfying $p - 1 \mid 2n$.

von Staudt-Clausen: $pB_{k(p-1)} \equiv -1 \pmod p$ $(k \geq 1)$.

Kummer (1850): If p is an odd prime and $p-1 \nmid b$, then

$$\frac{B_{k(p-1)+b}}{k(p-1)+b} \equiv \frac{B_b}{b} \pmod{p}.$$

The Euler numbers $\{E_n\}$ are given by

$$E_{2n-1} = 0,\ E_0 = 1,\ \sum_{r=0}^{n} \binom{2n}{2r} E_{2r} = 0\ (n \geq 1).$$

The first few Euler numbers are shown below:

$$E_0 = 1,\ E_2 = -1,\ E_4 = 5,\ E_6 = -61,$$
$$E_8 = 1385,\ E_{10} = -50521, E_{12} = 2702765.$$

The sequence $\{U_n\}$ is defined by

$$U_0 = 1 \quad \text{and} \quad U_n = -2\sum_{k=1}^{[n/2]}\binom{n}{2k}U_{n-2k}\ (n \geq 1).$$

Clearly $U_{2n-1} = 0$ for $n \geq 1$.

§ 2. p-regular functions

Definition 2.1 Let p be a prime. If $f(0), f(1), f(2), \ldots$ are all algebraic numbers which are integral for p, and for $n = 1, 2, 3, \ldots$,

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k f(k) \equiv 0 \pmod{p^n},$$

we call that f is a p-regular function.

Example 2.1 Let p be a prime, $b \in \{0, 1, 2, \ldots\}$ and $m \in \mathbb{N}$ with $p \nmid m$. Then

$$f(k) = m^{k(p-1)+b} \quad \text{and} \quad g(k) = m^{k(p-1)+b} - 1$$

are p-regular functions.

(by Fermat's little theorem and the binomial theorem)

Example 2.2 ([S1]) Let p be an odd prime and $b \in \{0, 1, 2, \ldots\}$. Then

$$f(k) = p(p - p^{k(p-1)+b})B_{k(p-1)+b}$$

is a p-regular function.

Example 2.3 ([S2]) Let p be an odd prime, $b \in \mathbb{N}$ and $p - 1 \nmid b$. Then

$$f(k) = (1 - p^{k(p-1)+b-1})\frac{B_{k(p-1)+b}}{k(p-1)+b}$$

is a p-regular function.

Example 2.4 ([S3]) Let p be an odd prime and $b \in \{0, 2, 4, \ldots\}$. Then

$$f(k) = \Big(1 - (-1)^{\frac{p-1}{2}}p^{k(p-1)+b}\Big)E_{k(p-1)+b}$$

is a p-regular function.

Example 2.5 ([S6]) Let p be an odd prime and $b \in \{0, 2, 4, \ldots\}$. Then

$$f(k) = \Big(1 - (\frac{p}{3})p^{k(p-1)+b}\Big)U_{k(p-1)+b}$$

is a p-regular function.

Lemma 2.1 Let $n \geq 1$ and $k \geq 0$ be integers. For any function f,

$$f(k) = \sum_{r=0}^{n-1} (-1)^{n-1-r} \binom{k-1-r}{n-1-r} \binom{k}{r} f(r) + \sum_{r=n}^{k} \binom{k}{r} (-1)^r \sum_{s=0}^{r} \binom{r}{s} (-1)^s f(s).$$

Proof. As $\sum_{j=0}^{m} (-1)^j \binom{x}{j} = (-1)^m \binom{x-1}{m}$, we find

$$\begin{aligned}
&\sum_{r=0}^{n-1} \binom{k}{r} (-1)^r \sum_{s=0}^{r} \binom{r}{s} (-1)^s f(s) \\
&= \sum_{s=0}^{n-1} \sum_{r=s}^{n-1} \binom{k}{r} \binom{r}{s} (-1)^{r-s} f(s) \\
&= \sum_{s=0}^{n-1} \binom{k}{s} \sum_{r=s}^{n-1} \binom{k-s}{r-s} (-1)^{r-s} f(s) \\
&= \sum_{s=0}^{n-1} \binom{k}{s} f(s) \sum_{j=0}^{n-1-s} \binom{k-s}{j} (-1)^j \\
&= \sum_{r=0}^{n-1} (-1)^{n-1-r} \binom{k-1-r}{n-1-r} \binom{k}{r} f(r).
\end{aligned}$$

By the binomial inversion formula,

$$f(k) = \sum_{r=0}^{k} \binom{k}{r}(-1)^r \sum_{s=0}^{r} \binom{r}{s}(-1)^s f(s).$$

Thus the result follows.

Theorem 2.1 Let p be a prime, $n \in \mathbb{N}$, $k \in \{0, 1, 2, \ldots\}$ and let f be a p-regular function. Then

$$f(k) \equiv \sum_{r=0}^{n-1} (-1)^{n-1-r} \binom{k-1-r}{n-1-r} \binom{k}{r} f(r) \pmod{p^n}.$$

Example: Let p be an odd prime, $k \in \{0, 1, 2, \ldots\}$, $n, b \in \mathbb{N}$ and $p - 1 \nmid b$. Then

$$(1 - p^{k(p-1)+b-1}) \frac{B_{k(p-1)+b}}{k(p-1)+b}$$
$$\equiv \sum_{r=0}^{n-1} (-1)^{n-1-r} \binom{k-1-r}{n-1-r} \binom{k}{r}$$
$$\times (1 - p^{r(p-1)+b-1}) \frac{B_{r(p-1)+b}}{r(p-1)+b} \pmod{p^n}.$$

Using the properties of Stirling numbers we deduce that:

Theorem 2.2 Let p be a prime. Then f is a p-regular function if and only if for each positive integer n, there are $a_0, a_1, \ldots, a_{n-1} \in \{0, 1, \ldots, p^n - 1\}$ such that

$$f(k) \equiv a_{n-1}k^{n-1} + \cdots + a_1 k + a_0 \pmod{p^n}$$

for every $k = 0, 1, 2, \ldots$. Moreover, we may assume $a_s \cdot s!/p^s \in \mathbb{Z}_p$ for $s = 0, 1, \ldots, n-1$. If $p \geq n$ and f is a p-regular function, then $a_0, \ldots, a_{n-1}$ are unique.

Example: For $k \in \mathbb{N}$,

$$\frac{B_{4k+2}}{4k+2}$$
$$\equiv 625k^4 + 875k^3 - 700k^2 + 180k - 1042 \pmod{5^5},$$
$$E_{4k} \equiv -750k^3 + 1375k^2 - 620k \pmod{5^5}\ (k > 1).$$

Lemma 2.2. *Let p be a prime. Let f be a p-regular function. Suppose $m, n \in \mathbb{N}$ and $t \in \mathbb{Z}$ with $t \geq 0$. Then*

$$\sum_{r=0}^{n} \binom{n}{r} (-1)^r f(p^{m-1} rt) \equiv 0 \pmod{p^{mn}}.$$

Moreover, if $A_k = p^{-k} \sum_{r=0}^{k} \binom{k}{r} (-1)^r f(r)$, then

$$\sum_{r=0}^{n} \binom{n}{r} (-1)^r f(p^{m-1} rt)$$
$$\equiv \begin{cases} p^{mn} t^n A_n \pmod{p^{mn+1}} \\ \quad \text{if } p > 2 \text{ or } m = 1, \\ 2^{mn} t^n \sum_{r=0}^{n} \binom{n}{r} A_{r+n} \pmod{2^{mn+1}} \\ \quad \text{if } p = 2 \text{ and } m \geq 2. \end{cases}$$

Theorem 2.3. *Let p be a prime, $k, m, n, t \in \mathbb{N}$, and let f be a p-regular function. Then*

$$f(ktp^{m-1}) \equiv \sum_{r=0}^{n-1} (-1)^{n-1-r} \binom{k-1-r}{n-1-r} \binom{k}{r} f(rtp^{m-1}) \pmod{p^{mn}}.$$

Moreover, setting $A_s = p^{-s} \sum_{r=0}^{s} \binom{s}{r} (-1)^r f(r)$ we then have

$$f(ktp^{m-1}) - \sum_{r=0}^{n-1} (-1)^{n-1-r} \binom{k-1-r}{n-1-r} \binom{k}{r} f(rtp^{m-1})$$
$$\equiv \begin{cases} p^{mn} \binom{k}{n} (-t)^n A_n \pmod{p^{mn+1}} \\ \quad \text{if } p > 2 \text{ or } m = 1, \\ 2^{mn} \binom{k}{n} (-t)^n \sum_{r=0}^{n} \binom{n}{r} A_{r+n} \pmod{2^{mn+1}} \\ \quad \text{if } p = 2 \text{ and } m \geq 2. \end{cases}$$

From Theorem 2.3 we deduce:

Theorem 2.4. *Let p be a prime, $k, m, t \in \mathbb{N}$. Let f be a p-regular function. Then*

(i) ([S2]) $f(kp^{m-1}) \equiv f(0) \pmod{p^m}$.

(ii) $f(ktp^{m-1}) \equiv kf(tp^{m-1}) - (k-1)f(0) \pmod{p^{2m}}$.

(iii) *We have*

$$f(ktp^{m-1}) \equiv \frac{k(k-1)}{2} f(2tp^{m-1}) - k(k-2)f(tp^{m-1}) + \frac{(k-1)(k-2)}{2} f(0) \pmod{p^{3m}}.$$

(iv) *We have*

$$f(kp^{m-1}) \equiv \begin{cases} f(0) - k(f(0) - f(1))p^{m-1} \pmod{p^{m+1}} \\ \quad \text{if } p > 2 \text{ or } m = 1, \\ f(0) - 2^{m-2}k(f(2) - 4f(1) + 3f(0)) \pmod{2^{m+1}} \\ \quad \text{if } p = 2 \text{ and } m \geq 2. \end{cases}$$

Example:

$$E_{k\varphi(p^m)+b} \equiv (1-kp^{m-1})(1-(-1)^{\frac{p-1}{2}}p^b)E_b + kp^{m-1}E_{p-1+b} \pmod{p^{m+1}},$$

$$U_{k\varphi(p^m)+b} \equiv \Big(1-(\frac{p}{3})p^b\Big)U_b \pmod{p^m},$$

where φ is Euler's totient function.

Lemma 2.3. *For $n = 0, 1, 2, \ldots$ and any two functions f and g we have*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k f(k)g(k) = \sum_{s=0}^{n}\binom{n}{s}\left(\sum_{r=0}^{s}\binom{s}{r}(-1)^r F(n-s+r)\right)G(s),$$

where $F(m) = \sum_{k=0}^{m}\binom{m}{k}(-1)^k f(k)$ and $G(m) = \sum_{k=0}^{m}\binom{m}{k}(-1)^k g(k)$.

Proof. We first claim that

$$\sum_{r=0}^{n}\binom{n}{r}(-1)^r f(r+m) = \sum_{r=0}^{m}\binom{m}{r}(-1)^r F(r+n).$$

Clearly the assertion holds for $m = 0$. Now assume that it is true for $m = k$. It is easily seen that

$$\sum_{r=0}^{n} \binom{n}{r}(-1)^r f(r+k+1)$$
$$= \sum_{s=0}^{n} \binom{n}{s}(-1)^s f(k+s) - \sum_{s=0}^{n+1} \binom{n+1}{s}(-1)^s f(k+s)$$
$$= \sum_{s=0}^{k} \binom{k}{s}(-1)^s F(n+s) - \sum_{s=0}^{k} \binom{k}{s}(-1)^s F(n+1+s)$$
$$= \sum_{s=0}^{k+1} \binom{k+1}{s}(-1)^s F(n+s).$$

So the assertion is true by induction.

From the binomial inversion formula we know that $g(k) = \sum_{s=0}^{k} \binom{k}{s}(-1)^s G(s)$. Thus, by the above assertion we have

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}(-1)^k f(k)g(k)\\
&=\sum_{k=0}^{n}\binom{n}{k}(-1)^k f(k)\sum_{s=0}^{k}\binom{k}{s}(-1)^s G(s)\\
&=\sum_{s=0}^{n}\left(\sum_{k=s}^{n}\binom{n}{k}\binom{k}{s}(-1)^{k-s}f(k)\right)G(s)\\
&=\sum_{s=0}^{n}\binom{n}{s}\left(\sum_{k=s}^{n}\binom{n-s}{k-s}(-1)^{k-s}f(k)\right)G(s)\\
&=\sum_{s=0}^{n}\binom{n}{s}\left(\sum_{r=0}^{n-s}\binom{n-s}{r}(-1)^{r}f(r+s)\right)G(s)\\
&=\sum_{s=0}^{n}\binom{n}{s}\left(\sum_{r=0}^{s}\binom{s}{r}(-1)^{r}F(n-s+r)\right)G(s),
\end{aligned}$$

which completes the proof.

Theorem 2.5 (Product Theorem)**.** *Let p be a prime. If f and g are p-regular functions, then $f \cdot g$ is also a p-regular function.*

§ 3. p-regular functions involving Bernoulli polynomials and generalized Bernoulli numbers

The Bernoulli polynomial $B_n(x)$ is given by

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}.$$

For $x \in \mathbb{Z}_p$ let $\langle x \rangle_p$ denote the unique number $n \in \{0, 1, \ldots, p-1\}$ such that $x \equiv n \pmod{p}$.

Theorem 3.1 Let p be a prime and let b be a nonnegative integer.

(i) ([S2, 2000], [Young, 2001]) If $p-1 \nmid b$, $x \in \mathbb{Z}_p$ and $x' = (x + \langle -x \rangle_p)/p$, then

$$f(k) = \frac{B_{k(p-1)+b}(x) - p^{k(p-1)+b-1} B_{k(p-1)+b}(x')}{k(p-1)+b}$$

is a $p-$regular function.

(ii) ([S2, (3.1), Theorem 3.1 and Remark 3.1]) If $a, b \in \mathbb{N}$ and $p \nmid a$, then

$$f(k) = (1-p^{k(p-1)+b-1})(a^{k(p-1)+b}-1)\frac{B_{k(p-1)+b}}{k(p-1)+b}$$

is a p-regular function.

Let χ be a primitive Dirichlet character of conductor m. The generalized Bernoulli number $B_{n,\chi}$ is defined by

$$\sum_{r=1}^{m} \frac{\chi(r)t\mathrm{e}^{rt}}{\mathrm{e}^{mt}-1} = \sum_{n=0}^{\infty} B_{n,\chi}\frac{t^n}{n!}.$$

Let χ_0 be the trivial character. It is well known that

$$B_{1,\chi_0} = \frac{1}{2},\ B_{n,\chi_0} = B_n\ (n \neq 1),$$
$$B_{n,\chi} = m^{n-1}\sum_{r=1}^{m} \chi(r)B_n\Big(\frac{r}{m}\Big).$$

If χ is nontrivial and $n \in \mathbb{N}$, then clearly $\sum_{r=1}^{m} \chi(r) = 0$ and so

$$\frac{B_{n,\chi}}{n} = m^{n-1}\sum_{r=1}^{m}\chi(r)\Big(\frac{B_n(\frac{r}{m}) - B_n}{n} + \frac{B_n}{n}\Big)$$
$$= m^{n-1}\sum_{r=1}^{m}\chi(r)\frac{B_n(\frac{r}{m}) - B_n}{n}.$$

When p is a prime with $p \nmid m$, by [S1, Lemma 2.3] we have $(B_n(\frac{r}{m}) - B_n)/n \in \mathbb{Z}_p$. Thus $B_{n,\chi}/n$ is congruent to an algebraic integer modulo p.

Theorem 3.2. *Let p be a prime and let b be a nonnegative integer.*

(i) ([Young, 1999], [Fox, 2002], [S2, 2000]) *If $b, m \in \mathbb{N}$, $p \nmid m$ and χ is a nontrivial primitive Dirichlet character of conductor m, then*

$$f(k) = (1 - \chi(p)p^{k(p-1)+b-1})\frac{B_{k(p-1)+b,\chi}}{k(p-1)+b}$$

is a $p-$regular function.

(ii) ([S2, Lemma 8.1(b)]) If $m \in \mathbb{N}$, $p \nmid m$ and χ is a nontrivial Dirichlet character of conductor m, then

$$f(k) = (1 - \chi(p)p^{k(p-1)+b-1})pB_{k(p-1)+b,\chi}$$

is a $p-$regular function.

Definition 3.1 For $a \neq 0$ define $\{E_n^{(a)}\}$ by

$$\sum_{k=0}^{[n/2]} \binom{n}{2k} a^{2k} E_{n-2k}^{(a)} = (1-a)^n \quad (n = 0, 1, 2, \ldots).$$

Clearly $E_n^{(1)} = E_n$.

Theorem 3.3 ([S6]). *Let a be a nonzero integer and $b \in \{0, 1, 2, \ldots\}$. Then $f(k) = E_{2k+b}^{(a)}$ is a $2-$regular function.*

Theorem 3.4 ([S6]). *Let p be an odd prime and let b be a nonnegative integer. Then*

$$f_2(k) = (1-(-1)^{\frac{p-1}{2}b+[\frac{p-1}{4}]}p^{k(p-1)+b})E^{(2)}_{k(p-1)+b}$$

and

$$f_3(k) = (1-(-1)^{[\frac{p+1}{6}]}(\frac{p}{3})^{b+1}p^{k(p-1)+b})E^{(3)}_{k(p-1)+b}$$

are p-regular functions.

§4. Congruences for $\sum_{x=1}^{p-1}\frac{1}{x^k} \pmod{p^3}$ and $\sum_{x=1}^{\frac{p-1}{2}}\frac{1}{x^k} \pmod{p^3}$

Theorem 4.1 ([S2]) Let p be a prime greater than 3.

(a) If $k \in \{1, 2, \ldots, p-4\}$, then

$$\sum_{x=1}^{p-1} \frac{1}{x^k} \equiv \begin{cases} \frac{k(k+1)}{2} \frac{B_{p-2-k}}{p-2-k} p^2 \pmod{p^3} \\ \quad \text{if } k \text{ is odd,} \\ k\left(\frac{B_{2p-2-k}}{2p-2-k} - 2\frac{B_{p-1-k}}{p-1-k}\right) p \pmod{p^3} \\ \quad \text{if } k \text{ is even.} \end{cases}$$

(b) $$\sum_{x=1}^{p-1} \frac{1}{x^{p-3}} \equiv (\frac{1}{2} - 3B_{p+1})p - \frac{4}{3}p^2 \pmod{p^3}.$$

(c) $$\sum_{x=1}^{p-1} \frac{1}{x^{p-2}} \equiv -(2 + pB_{p-1})p + \frac{5}{2}p^2 \pmod{p^3}.$$

(d) $$\sum_{x=1}^{p-1} \frac{1}{x^{p-1}} \equiv pB_{2p-2} - 3pB_{p-1} + 3(p-1) \pmod{p^3}.$$

Proof. For $m \in \mathbb{Z}$ it is clear that

$$1^m + 2^m + \cdots + (p-1)^m = \frac{B_{m+1}(p) - B_{m+1}}{m+1}$$

$$= \frac{1}{m+1} \sum_{r=1}^{m+1} \binom{m+1}{r} B_{m+1-r} p^r$$

$$= pB_m + \frac{p^2}{2} m B_{m-1}$$

$$+ \sum_{r=3}^{m+1} \binom{m}{r-1} pB_{m+1-r} \cdot \frac{p^{r-4}}{r} \cdot p^3.$$

Since $pB_{m+1-r}, \frac{p^{r-4}}{r} \in \mathbb{Z}_p$ for $r \geq 3$ we have

(4.1)

$$1^m + 2^m + \cdots + (p-1)^m \equiv pB_m + \frac{p^2}{2} m B_{m-1} \pmod{p^3}.$$

Let $k \in \{1, 2, \ldots, p-1\}$. From (4.1) and Euler's theorem we see that

$$\sum_{x=1}^{p-1} \frac{1}{x^k} \equiv \sum_{x=1}^{p-1} x^{\varphi(p^3)-k}$$

$$\equiv pB_{\varphi(p^3)-k} + \frac{p^2}{2}\left(\varphi(p^3)-k\right)B_{\varphi(p^3)-k-1}$$

$$\equiv pB_{\varphi(p^3)-k} - \frac{k}{2}p^2 B_{\varphi(p^3)-k-1}$$

$$= \begin{cases} pB_{\varphi(p^3)-k} \pmod{p^3} & \text{if } k \text{ is even,} \\ -\frac{k}{2}p^2 B_{\varphi(p^3)-k-1} \pmod{p^3} & \text{if } k \text{ is odd.} \end{cases}$$

For $k \in \{1, 2, \ldots, p-2\}$ we see that

$$\frac{B_{\varphi(p^3)-k}}{\varphi(p^3)-k} = \frac{B_{(p^2-1)(p-1)+p-1-k}}{(p^2-1)(p-1)+p-1-k}$$

$$\equiv (p^2-1)\frac{B_{2p-2-k}}{2p-2-k} - (p^2-2)(1-p^{p-2-k})\frac{B_{p-1-k}}{p-1-k}$$

$$\equiv -\frac{B_{2p-2-k}}{2p-2-k} + 2(1-p^{p-2-k})\frac{B_{p-1-k}}{p-1-k} \pmod{p^2}.$$

Thus,

(4.2)

$$pB_{\varphi(p^3)-k}$$
$$\equiv -kp\Big(-\frac{B_{2p-2-k}}{2p-2-k} + 2(1-p^{p-2-k})\frac{B_{p-1-k}}{p-1-k}\Big)$$
$$\equiv \begin{cases} kp(\frac{B_{2p-2-k}}{2p-2-k} - 2\frac{B_{p-1-k}}{p-1-k}) \pmod{p^3} \\ \qquad \text{if } k < p-3, \\ (p-3)p(\frac{B_{p+1}}{p+1} - 2(1-p)\frac{B_2}{2}) \pmod{p^3} \\ \qquad \text{if } k = p-3. \end{cases}$$

When $k \in \{1, 2, \ldots, p{-}3\}$, it follows from Kummer's congruences that

$$\frac{B_{\varphi(p^3)-k-1}}{\varphi(p^3)-k-1} = \frac{B_{(p^2-1)(p-1)+p-2-k}}{(p^2-1)(p-1)+p-2-k}$$
$$\equiv \frac{B_{p-2-k}}{p-2-k} \pmod{p}.$$

Thus,

(4.3)

$$-\frac{k}{2}p^2B_{\varphi(p^3)-k-1} \equiv -\frac{k}{2}p^2(-k-1)\frac{B_{p-2-k}}{p-2-k} \pmod{p^3}.$$

Combining the above we get

$$\sum_{x=1}^{p-1}\frac{1}{x^k}\equiv\begin{cases} kp\left(\dfrac{B_{2p-2-k}}{2p-2-k}-2\dfrac{B_{p-1-k}}{p-1-k}\right) \pmod{p^3} \\ \qquad \text{if } k\in\{2,4,\dots,p-5\}, \\ (\dfrac{1}{2}-3B_{p+1})p-\dfrac{4}{3}p^2 \pmod{p^3} \\ \qquad \text{if } k=p-3, \\ \dfrac{k(k+1)}{2}\dfrac{B_{p-2-k}}{p-2-k}p^2 \pmod{p^3} \\ \qquad \text{if } k\in\{1,3,\dots,p-4\}. \end{cases}$$

This proves parts (a) and (b).

Now consider parts (c) and (d). Note that $pB_{r(p-1)}\equiv -1 \pmod p$ for $r\ge 1$. From the above and [S1, Corollary 4.2] we see that

$$\begin{aligned}\sum_{x=1}^{p-1}\frac{1}{x^{p-2}} &\equiv -\frac{p-2}{2}p^2B_{\varphi(p^3)-(p-1)} \\ &\equiv -\frac{p-2}{2}p((p^2-1)pB_{p-1}-(p^2-2)(p-1)) \\ &\equiv \frac{p-2}{2}p(pB_{p-1}+2-2p) \\ &\equiv -p(pB_{p-1}+2)+\frac{5}{2}p^2 \pmod{p^3}\end{aligned}$$

and

$$
\begin{aligned}
&\sum_{x=1}^{p-1} \frac{1}{x^{p-1}} \\
&\equiv pB_{\varphi(p^3)-(p-1)} \\
&\equiv \binom{p^2-1}{2} pB_{2p-2} - (p^2-1)(p^2-3)pB_{p-1} \\
&\quad + \binom{p^2-2}{2}(p-1) \\
&\equiv (1 - \frac{3p^2}{2})pB_{2p-2} - (3-4p^2)pB_{p-1} \\
&\quad + (3 - \frac{5p^2}{2})(p-1) \\
&\equiv pB_{2p-2} - 3pB_{p-1} + 3(p-1) \pmod{p^3}.
\end{aligned}
$$

This concludes the proof.

One can similarly prove that

Theorem 4.2 Let $p > 5$ be a prime and $k \in \{1, 2, \ldots, p-5\}$. Then

$$\sum_{x=1}^{p-1} \frac{1}{x^k} \equiv \begin{cases} -k\left(\frac{B_{3p-3-k}}{3p-3-k} - 3\frac{B_{2p-2-k}}{2p-2-k} + 3\frac{B_{p-1-k}}{p-1-k}\right)p \\ \qquad -\binom{k+2}{3}\frac{p^3 B_{p-3-k}}{p-3-k} \pmod{p^4} \quad \text{if } 2|k, \\ -\binom{k+1}{2}\left(\frac{B_{2p-3-k}}{2p-3-k} - 2\frac{B_{p-2-k}}{p-2-k}\right)p^2 \pmod{p^4} \\ \qquad\qquad \text{if } 2 \nmid k. \end{cases}$$

Theorem 4.3 ([S2]). *Let $p > 5$ be a prime.*

(a) If $k \in \{2, 4, \ldots, p-5\}$, then

$$\sum_{x=1}^{\frac{p-1}{2}} \frac{1}{x^k}$$
$$\equiv \frac{k(2^{k+1}-1)}{2} p\left(\frac{B_{2p-2-k}}{2p-2-k} - 2\frac{B_{p-1-k}}{p-1-k}\right) \pmod{p^3}.$$

(b) If $k \in \{3, 5, \cdots, p-4\}$, then

$$\sum_{x=1}^{\frac{p-1}{2}} \frac{1}{x^k} \equiv (2^k - 2)\left(2\frac{B_{p-k}}{p-k} - \frac{B_{2p-1-k}}{2p-1-k}\right) \pmod{p^2}.$$

(c) If $q_p(2) = (2^{p-1}-1)/p$, then

$$\sum_{x=1}^{\frac{p-1}{2}} \frac{1}{x}$$
$$\equiv -2q_p(2) + pq_p^2(2) - \frac{2}{3}p^2 q_p^3(2) - \frac{7}{12}p^2 B_{p-3} \pmod{p^3}.$$

Theorem 4.4 ([S4]). *Let $p > 3$ be a prime and $q_p(a) = (a^{p-1} - 1)/p$. Then*

$$\sum_{\substack{k=1 \\ k\equiv p \pmod 3}}^{p-1} \frac{1}{k}$$

$$\equiv \frac{1}{2}q_p(3) - \frac{1}{4}pq_p(3)^2 + \frac{1}{6}p^2q_p(3)^3 - \frac{p^2}{81}B_{p-3} (\textit{mod } p^3$$

$$\sum_{\substack{k=1 \\ k\equiv p \pmod 4}}^{p-1} \frac{1}{k}$$

$$\equiv \frac{3}{4}q_p(2) - \frac{3}{8}pq_p(2)^2 + \frac{1}{4}p^2q_p(2)^3$$

$$- \frac{p^2}{192}B_{p-3} \pmod{p^3},$$

$$\sum_{\substack{k=1 \\ k\equiv p \pmod 6}}^{p-1} \frac{1}{k}$$

$$\equiv \frac{1}{3}q_p(2) + \frac{1}{4}q_p(3) - p\left(\frac{1}{6}q_p(2)^2 + \frac{1}{8}q_p(3)^2\right)$$

$$+ p^2\left(\frac{1}{9}q_p(2)^3 + \frac{1}{12}q_p(3)^3 - \frac{1}{648}B_{p-3}\right) \pmod{p^3}.$$

§5. **A congruence for** $(p-1)!$ (mod p^3)

Let p be a prime greater than 3. The classical Wilson's theorem states that

$$(p-1)! \equiv -1 \pmod{p}.$$

In 1900 J.W.L.Glaisher showed that

$$(p-1)! \equiv pB_{p-1} - p \pmod{p^2}.$$

Here we give a congruence for $(p-1)!$ modulo p^3.

Theorem 5.1 ([S2]). *For any prime $p > 3$ we have*

$$(p-1)! \equiv \frac{pB_{2p-2}}{2p-2} - \frac{pB_{p-1}}{p-1} - \frac{1}{2}\left(\frac{pB_{p-1}}{p-1}\right)^2 \pmod{p^3}.$$

The proof is based on the following Newton's formula.

Newton's formula: Suppose that $x_1, x_2, \ldots, x_n$ are complex numbers. If

$$S_m = x_1^m + x_2^m + \cdots + x_n^m,$$
$$A_m = \sum_{1 \le i_1 < i_2 < \cdots < i_m \le n} x_{i_1} x_{i_2} \cdots x_{i_m},$$

for $k = 0, 1, \ldots, n$ we have

$$S_k - A_1 S_{k-1} + A_2 S_{k-2} + \cdots + (-1)^{k-1} A_{k-1} S_1 + (-1)^k k A_k = 0.$$

§6. Congruences involving Bernoulli and Euler polynomials

The Euler polynomials $\{E_n(x)\}$ are given by

$$E_n(x) + \sum_{r=0}^{n} \binom{n}{r} E_r(x) = 2x^n \quad (n \ge 0).$$

It is known that

$$E_n(x) = \frac{1}{2^n} \sum_{r=0}^{n} \binom{n}{r} (2x-1)^{n-r} E_r$$
$$= \frac{2}{n+1} \Big(B_{n+1}(x) - 2^{n+1} B_{n+1}\Big(\frac{x}{2}\Big)\Big).$$

Also,

$$\sum_{x=0}^{p-1} x^k = \frac{B_{k+1}(p) - B_{k+1}}{k+1},$$

$$\sum_{x=0}^{p-1} (-1)^x x^k = -\frac{(-1)^p E_k(p) - E_k(0)}{2}.$$

Theorem 6.1 ([S3]). *Let* $p, m \in \mathbb{N}$ *and* $k, r \in \mathbb{Z}$ *with* $k \geq 0$. *Then*

$$\sum_{\substack{x=0 \\ x\equiv r \pmod{m}}}^{p-1} x^k$$

$$= \frac{m^k}{k+1}\left(B_{k+1}\left(\frac{p}{m} + \left\{\frac{r-p}{m}\right\}\right) - B_{k+1}\left(\left\{\frac{r}{m}\right\}\right)\right)$$

and

$$\sum_{\substack{x=0\\ x\equiv r(\text{mod } m)}}^{p-1} (-1)^{\frac{x-r}{m}} x^k$$
$$= -\frac{m^k}{2}\Big((-1)^{[\frac{r-p}{m}]}E_k\Big(\frac{p}{m}+\Big\{\frac{r-p}{m}\Big\}\Big) - (-1)^{[\frac{r}{m}]}E_k\Big(\Big\{\frac{r}{m}\Big\}\Big)\Big).$$

Proof. For any real number t and nonnegative integer n it is well known that

$$B_n(t+1) - B_n(t) = nt^{n-1} \ (n \neq 0),$$
$$E_n(t+1) + E_n(t) = 2t^n.$$

Hence, for $x \in \mathbb{Z}$ we have

$$B_{k+1}\Big(\frac{x+1}{m}+\Big\{\frac{r-x-1}{m}\Big\}\Big) - B_{k+1}\Big(\frac{x}{m}+\Big\{\frac{r-x}{m}\Big\}\Big)$$
$$= \begin{cases} 0 & \text{if } m \nmid x-r, \\ (k+1)(\frac{x}{m})^k & \text{if } m \mid x-r. \end{cases}$$

Thus

$$B_{k+1}\Big(\frac{p}{m}+\Big\{\frac{r-p}{m}\Big\}\Big)-B_{k+1}\Big(\Big\{\frac{r}{m}\Big\}\Big)$$
$$=\sum_{x=0}^{p-1}\Big(B_{k+1}\Big(\frac{x+1}{m}+\Big\{\frac{r-x-1}{m}\Big\}\Big)$$
$$-B_{k+1}\Big(\frac{x}{m}+\Big\{\frac{r-x}{m}\Big\}\Big)\Big)$$
$$=\frac{k+1}{m^k}\sum_{\substack{x=0\\ x\equiv r \pmod{m}}}^{p-1} x^k.$$

Similarly, if $x\in\mathbb{Z}$, then

$$(-1)^{[\frac{r-x-1}{m}]}E_k\Big(\frac{x+1}{m}+\Big\{\frac{r-x-1}{m}\Big\}\Big)$$
$$-(-1)^{[\frac{r-x}{m}]}E_k\Big(\frac{x}{m}+\Big\{\frac{r-x}{m}\Big\}\Big)$$
$$=\begin{cases}0 & \text{if } m\nmid x-r,\\ -(-1)^{\frac{r-x}{m}}\cdot 2(\frac{x}{m})^k & \text{if } m\mid x-r.\end{cases}$$

Thus

$$(-1)^{[\frac{r-p}{m}]}E_k\Big(\frac{p}{m}+\Big\{\frac{r-p}{m}\Big\}\Big)$$
$$-(-1)^{[\frac{r}{m}]}E_k\Big(\Big\{\frac{r}{m}\Big\}\Big)$$
$$=\sum_{x=0}^{p-1}\Big\{(-1)^{[\frac{r-x-1}{m}]}E_k\Big(\frac{x+1}{m}+\Big\{\frac{r-x-1}{m}\Big\}\Big)$$
$$-(-1)^{[\frac{r-x}{m}]}E_k\Big(\frac{x}{m}+\Big\{\frac{r-x}{m}\Big\}\Big)\Big\}$$
$$=-\frac{2}{m^k}\sum_{\substack{x=0\\ x\equiv r \pmod m}}^{p-1}(-1)^{\frac{x-r}{m}}x^k.$$

This completes the proof.

Corollary 6.1 Let p be an odd prime and $k\in\{0,1,\ldots,p-2\}$. Let $r\in\mathbb{Z}$ and $m\in\mathbb{N}$ with $p\nmid m$. Then

$$\sum_{\substack{x=0\\ x\equiv r \pmod m}}^{p-1}x^k$$
$$\equiv\frac{m^k}{k+1}\Big(B_{k+1}\Big(\Big\{\frac{r-p}{m}\Big\}\Big)-B_{k+1}\Big(\Big\{\frac{r}{m}\Big\}\Big)\Big) \pmod p$$

and

$$\sum_{\substack{x=0\\ x\equiv r(\text{mod } m)}}^{p-1} (-1)^{\frac{x-r}{m}} x^k$$
$$\equiv -\frac{m^k}{2}\Big((-1)^{[\frac{r-p}{m}]} E_k\Big(\Big\{\frac{r-p}{m}\Big\}\Big) - (-1)^{[\frac{r}{m}]} E_k\Big(\Big\{\frac{r}{m}\Big\}\Big)\Big) (\text{mod } p).$$

Proof. If $x_1, x_2 \in \mathbb{Z}_p$ and $x_1 \equiv x_2$ (mod p), then $\frac{B_{k+1}(x_1)-B_{k+1}(x_2)}{k+1} \equiv 0$ (mod p) and $E_k(x_1) \equiv E_k(x_2)$ (mod p). Thus the result follows from Theorem 6.1.

In the case $k = p-2$, Corollary 6.1 is due to Zhi-Wei Sun. Inspired by Zhi-Wei Sun's work, I established Theorem 6.1 and Corollary 6.1.

Corollary 6.2 Let p be an odd prime. Let $k \in \{0, 1, \ldots, p-2\}$ and $m, s \in \mathbb{N}$ with $p \nmid m$. Then

$$\frac{(-1)^k}{k+1}\Big(B_{k+1}\Big(\Big\{\frac{(s-1)p}{m}\Big\}\Big) - B_{k+1}\Big(\Big\{\frac{sp}{m}\Big\}\Big)\Big) \equiv \sum_{\frac{(s-1)p}{m}<r\le\frac{sp}{m}} r^k \pmod p$$

and

$$(-1)^{[\frac{(s-1)p}{m}]}E_k\Big(\Big\{\frac{(s-1)p}{m}\Big\}\Big) - (-1)^{[\frac{sp}{m}]}E_k\Big(\Big\{\frac{sp}{m}\Big\}\Big) \equiv 2(-1)^{k-1}\sum_{\frac{(s-1)p}{m}<r\le\frac{sp}{m}} (-1)^r r^k \pmod p.$$

Theorem 6.2. *Let $m, s \in \mathbb{N}$ and let p be an odd prime not dividing m. Then*

$$(-1)^s \frac{m}{p} \sum_{\substack{k=1 \\ k \equiv sp (\text{mod } m)}}^{p-1} \binom{p}{k} \equiv \sum_{\frac{(s-1)p}{m} < k < \frac{sp}{m}} \frac{(-1)^{km}}{k}$$

$$\equiv \begin{cases} B_{p-1}(\{\frac{(s-1)p}{m}\}) - B_{p-1}(\{\frac{sp}{m}\}) (\text{mod } p) \text{if } 2 \mid m, \\ \frac{1}{2}((-1)^{[\frac{(s-1)p}{m}]} E_{p-2}(\{\frac{(s-1)p}{m}\}) \\ \quad -(-1)^{[\frac{sp}{m}]} E_{p-2}(\{\frac{sp}{m}\})) \ (\text{mod } p) \ \text{if } 2 \nmid m. \end{cases}$$

Corollary 6.3. *Let $m, n \in \mathbb{N}$ and let p be an odd prime not dividing m.*

(i) *If $2 \mid m$, then*

$$B_{p-1}\left(\left\{\frac{np}{m}\right\}\right) - B_{p-1}$$
$$\equiv \frac{m}{p} \sum_{s=1}^{n} (-1)^{s-1} \sum_{\substack{k=1 \\ k \equiv sp (\text{mod } m)}}^{p-1} \binom{p}{k} \ (\text{mod } p).$$

(ii) If $2 \nmid m$, then

$$(-1)^{[\frac{np}{m}]}E_{p-2}\Big(\Big\{\frac{np}{m}\Big\}\Big)+\frac{2^p-2}{p} \equiv \frac{2m}{p}\sum_{s=1}^{n}(-1)^{s-1}\sum_{\substack{k=1\\ k\equiv sp \pmod m}}^{p-1}\binom{p}{k} \pmod p.$$

In the cases $m=3,4,5,6,8,9$ the formulae for $\sum_{\substack{k=0\\ k\equiv r \pmod m}}^{p}\binom{p}{k}$ were given by me (published in 1992-1993), in the case $m=10$ the formula was published by Z.H.Sun and Z.W.Sun in 1992, and for the case $m=12$, the formula was given by Zhi-Wei Sun (published in 2002).

§7. Extension of Stern's congruence for Euler numbers

For $k, m \in \mathbb{N}$ and $b \in \{0, 2, 4, \ldots\}$. The Stern's congruence states that

$$(7.1) \qquad E_{2^m k+b} \equiv E_b + 2^m k \pmod{2^{m+1}}.$$

In 1875 Stern gave a brief sketch of a proof of (7.1). Then Frobenius amplified Stern's sketch in 1910.

There are many modern proofs of (7.1). (Ernvall(1979), Wagstaff(2002), Zhi-Wei Sun(2005), Zhi-Hong Sun(2008))

Let $b \in \{0, 2, 4, \ldots\}$ and $k, m \in \mathbb{N}$. In [S5, 2010] I showed that

$$E_{2^m k+b} \equiv E_b + 2^m k \pmod{2^{m+2}} \quad \text{for} \quad m \geq 2,$$
$$E_{2^m k+b} \equiv E_b + 5 \cdot 2^m k \pmod{2^{m+3}} \quad \text{for} \quad m \geq 3,$$

and for $m \geq 4$,

$$E_{2^m k+b} \equiv \begin{cases} E_b + 5\cdot 2^m k \pmod{2^{m+4}} & \text{if } b \equiv 0,6 \pmod 8, \\ E_b - 3\cdot 2^m k \pmod{2^{m+4}} & \text{if } b \equiv 2,4 \pmod 8. \end{cases}$$

In [S9], Z.H.Sun and L.L.Wang (IJNT, 2013) established a congruence for $E_{2^m k+b} \pmod{2^{m+7}}$. In particular, for $m \geq 7$,

$$E_{2^m k+b} \equiv E_b + 2^m k(7(b+1)^2 - 18) \pmod{2^{m+7}}.$$

For $a \neq 0$ recall that $\{E_n^{(a)}\}$ is defined by

$$\sum_{k=0}^{[n/2]} \binom{n}{2k} a^{2k} E_{n-2k}^{(a)} = (1-a)^n \quad (n = 0,1,2,\dots).$$

From [S6] we know that

$$E_n^{(a)} = (2a)^n E_n\Big(\frac{1}{2a}\Big) = \sum_{k=0}^{[n/2]} \binom{n}{2k} (1-a)^{n-2k} a^{2k} E_{2k}$$
$$= \sum_{k=0}^{n} \binom{n}{k} 2^{k+1} \Big(1 - 2^{k+1}\Big) \frac{B_{k+1}}{k+1} a^k.$$

Theorem 7.1([S6, 2012]). *Let a be a nonzero integer, $k, m \in \mathbb{N}$, $m \geq 2$ and $b \in \{0, 1, 2, \ldots\}$. Then*

$$E^{(a)}_{2^m k+b} - E^{(a)}_b$$
$$\equiv \begin{cases} 2^m k(a^3((b-1)^2+5) - a + 2^m k a^3(b-1)) \\ \qquad (\text{mod } 2^{m+4+3\alpha}) \text{ if } 2^\alpha \mid a \text{ and } \alpha \geq 1, \\ 2^m k a((b+1)^2 + 4 - 2^m k(b+1)) \pmod{2^{m+4}} \\ \qquad\qquad\qquad \text{if } 2 \nmid a \text{ and } 2 \mid b, \\ 2^m k(a^2-1) \pmod{2^{m+4}} \quad \text{if } 2 \nmid ab. \end{cases}$$

As $E^{(1)}_n = E_n$, the theorem is a vast generalization of Stern's congruence.

§8. $(-1)^n U_{2n}$ and $(-1)^n E_{2n}^{(a)}$ are realizable

If $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences satisfying

$$a_1 = b_1, \quad b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 = n a_n \ (n > 1),$$

we say that (a_n, b_n) is a Newton-Euler pair. If (a_n, b_n) is a Newton-Euler pair and $a_n \in \mathbb{Z}$ for all $n = 1, 2, 3, \ldots$, we say that $\{b_n\}$ is a Newton-Euler sequence.

Let $\{b_n\}$ be a Newton-Euler sequence. Then clearly $b_n \in \mathbb{Z}$ for all $n = 1, 2, 3, \ldots$.

Z.H. Sun, On the properties of Newton-Euler pairs, J. Number Theory 114(2005), 88-123.

Lemma 8.1. *Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of integers. Then the following statements are equivalent:*

(i) $\{b_n\}$ is a Newton-Euler sequence.

(ii) $\sum_{d|n}\mu(\frac{n}{d})b_d \equiv 0 \pmod{n}$ for every $n \in \mathbb{N}$.

(iii) For any prime p and $\alpha, m \in \mathbb{N}$ with $p \nmid m$ we have $b_{mp^\alpha} \equiv b_{mp^{\alpha-1}} \pmod{p^\alpha}$.

(iv) For any $n, t \in \mathbb{N}$ and prime p with $p^t \parallel n$ we have $b_n \equiv b_{\frac{n}{p}} \pmod{p^t}$.

(v) There exists a sequence $\{c_n\}$ of integers such that $b_n = \sum_{d|n} dc_d$ for any $n \in \mathbb{N}$.

(vi) For any $n \in \mathbb{N}$ we have

$$\sum_{k_1+2k_2+\cdots+nk_n=n} \frac{b_1^{k_1} b_2^{k_2} \cdots b_n^{k_n}}{1^{k_1} \cdot k_1! \cdot 2^{k_2} \cdot k_2! \cdots n^{k_n} \cdot k_n!} \in \mathbb{Z}.$$

(vii) For any $n \in \mathbb{N}$ we have

$$\frac{1}{n!}\begin{vmatrix} b_1 & b_2 & b_3 & \dots & b_n \\ -1 & b_1 & b_2 & \dots & b_{n-1} \\ & -2 & b_1 & \dots & b_{n-2} \\ & & \ddots & \ddots & \vdots \\ & & & -(n-1) & b_1 \end{vmatrix} \in \mathbb{Z}.$$

Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of nonnegative integers. If there is a set X and a map $T: X \to X$ such that b_n is the number of fixed points of T^n, following Puri and Ward we say that $\{b_n\}$ is realizable.

Puri and Ward (2001) proved that a sequence $\{b_n\}$ of nonnegative integers is realizable if and only if for any $n \in \mathbb{N}$, $\frac{1}{n}\sum_{d|n}\mu(\frac{n}{d})b_d$ is a nonnegative integer. Thus, using Möbius inversion formula we see that a sequence $\{b_n\}$ is realizable if and only if there exists a sequence $\{c_n\}$ of nonnegative integers such that $b_n = \sum_{d|n} dc_d$ for any $n \in \mathbb{N}$.

J. Arias de Reyna (Acta Arith. 119(2005), 39-52) showed that $\{E_{2n}\}$ is a Newton-Euler sequence and $\{|E_{2n}|\}$ is realizable.

In [S6, S7] I proved the following results.

Theorem 8.1 ([S7, 2012]). $\{U_{2n}\}$ *is a Newton-Euler sequence and* $\{(-1)^n U_{2n}\}$ *is realizable.*

Theorem 8.2 ([S6, 2012]). *Let* $a \in \mathbb{N}$. *Then* $\{(-1)^n E_{2n}^{(a)}\}$ *is realizable.*

§9. **Congruences involving** $\{U_n\}$

In [S7], Z.H. Sun introduced the sequence $\{U_n\}$ *as below:*

$$U_0 = 1, \quad U_n = -2 \sum_{k=1}^{[n/2]} \binom{n}{2k} U_{n-2k} \quad (n \geq 1).$$

Since $U_1 = 0$. *By induction,* $U_{2n-1} = 0$ *for* $n \geq 1$.

The first few values of U_{2n} are shown below:

$U_2 = -2, \quad U_4 = 22, \quad U_6 = -602, \quad U_8 = 30742,$
$U_{10} = -2523002, \quad U_{12} = 303692662.$

Theorem 9.1 For $n \in \mathbb{N}$ we have

$$U_{2n} = 3^{2n} E_{2n}\Big(\frac{1}{3}\Big) = -2\Big(2^{2n+1}+1\Big)3^{2n}\frac{B_{2n+1}(\frac{1}{3})}{2n+1}$$
$$= -\frac{2(2^{2n+1}+1)6^{2n}}{2^{2n}+1} \cdot \frac{B_{2n+1}(\frac{1}{6})}{2n+1}.$$

For $d \in \mathbb{Z}$ with $d < 0$ and $d \equiv 0, 1 \pmod 4$ let $h(d)$ denote the class number of the form class group consisting of classes of primitive, integral binary quadratic forms of discriminant d.

Theorem 9.2 ([S7]). *Let p be a prime of the form $4k+1$. Then*

$$U_{\frac{p-1}{2}} \equiv \Big(1 + 2(-1)^{\frac{p-1}{4}}\Big)h(-3p) \pmod p$$

and so $p \nmid U_{\frac{p-1}{2}}$.

Recall that the Fermat quotient $q_p(a) = (a^{p-1} - 1)/p$.

Theorem 9.3 ([S7]). *Let* p *be a prime greater than* 5. *Then*

(i) $\sum_{k=1}^{[p/6]} \frac{1}{k} \equiv -2q_p(2) - \frac{3}{2}q_p(3) + p\Big(q_p(2)^2 + \frac{3}{4}q_p(3)^2\Big) - \frac{5p}{2}(\frac{p}{3})U_{p-3} \pmod{p^2}$,

(ii) $\sum_{k=1}^{[p/3]} \frac{1}{k} \equiv -\frac{3}{2}q_p(3) + \frac{3}{4}pq_p(3)^2 - p(\frac{p}{3})U_{p-3} \pmod{p^2}$,

(iii) $\sum_{k=1}^{[2p/3]} \frac{(-1)^{k-1}}{k} \equiv 9 \sum_{\substack{k=1 \\ 3|k+p}}^{p-1} \frac{1}{k} \equiv 3p(\frac{p}{3})U_{p-3} \pmod{p^2}$.

(iv) We have

$$(-1)^{[\frac{p}{6}]}\binom{p-1}{[\frac{p}{6}]}$$
$$\equiv 1+p\Big(2q_p(2)+\frac{3}{2}q_p(3)\Big)+p^2\Big(q_p(2)^2+3q_p(2)q_p(3) + \frac{3}{8}q_p(3)^2-5\Big(\frac{p}{3}\Big)U_{p-3}\Big) \pmod{p^3}$$

and

$$(-1)^{[\frac{p}{3}]}\binom{p-1}{[\frac{p}{3}]}$$
$$\equiv 1+\frac{3}{2}pq_p(3)+\frac{3}{8}p^2q_p(3)^2-\frac{p^2}{2}\Big(\frac{p}{3}\Big)U_{p-3} \pmod{p^3}.$$

Theorem 9.4. *Let $p>3$ be a prime and $k\in\{2,4,\dots,p-3\}$. Then*

$$\sum_{x=1}^{[p/6]}\frac{1}{x^k}\equiv 6^k\sum_{\substack{x=1\\6|x-p}}^{p-1}\frac{1}{x^k}\equiv\frac{6^k(2^k+1)}{4(2^{k-1}+1)}\Big(\frac{p}{3}\Big)U_{p-1-k} \pmod{}$$

and

$$\sum_{x=1}^{[p/3]} \frac{1}{x^k} \equiv 3^k \sum_{\substack{x=1 \\ 3|x-p}}^{p-1} \frac{1}{x^k} \equiv \frac{6^k}{4(2^{k-1}+1)} \left(\frac{p}{3}\right) U_{p-1-k} \ (\mathrm{mod}$$

Theorem 9.5. *Let $p > 3$ be a prime and $k \in \{2, 4, \ldots, p-3\}$. Then*

$$\sum_{x=1}^{[p/3]} (-1)^{x-1} \frac{1}{x^k} \equiv -\frac{3^k}{2} \left(\frac{p}{3}\right) U_{p-1-k} \pmod p$$

and

$$\sum_{x=1}^{[\frac{p+3}{6}]} \frac{1}{(2x-1)^k} \equiv -\frac{3^k}{2^{k+1}+4} \left(\frac{p}{3}\right) U_{p-1-k} \pmod p.$$

By [S7],

(9.1)

$$B_{p-2}\left(\frac{1}{3}\right) \equiv 6U_{p-3} \pmod{p} \quad \text{for any prime} \quad p > 3.$$

S. Mattarei and R. Tauraso (Congruences for central binomial sums and finite polylogarithms, J. Number Theory 133(2013), 131-157) proved that for any prime $p > 3$,

$$\sum_{k=0}^{p-1}\binom{2k}{k} \equiv \left(\frac{p}{3}\right) - \frac{p^2}{3}B_{p-2}\left(\frac{1}{3}\right) \pmod{p^3}.$$

Thus,

$$(9.2) \quad \sum_{k=0}^{p-1}\binom{2k}{k} \equiv \left(\frac{p}{3}\right) - 2p^2U_{p-3} \pmod{p^3}$$

for any prime $p > 3$. The congruence

$$\sum_{k=0}^{p-1}\binom{2k}{k} \equiv \left(\frac{p}{3}\right) \pmod{p^2}$$

was found and proved by Z.W. Sun and R. Tauraso (Adv. in Appl. Math. 45(2010),125-148).

Suppose that p is a prime of the form $3k+1$ and so $4p = L^2+27M^2$ for some integers L and M. Assume $L \equiv 1 \pmod 3$. From (9.1) and the work of J. B. Cosgrave and K. Dilcher (Mod p^3 analogues of theorems of Gauss and Jacobi on binomial coefficients, Acta Arith. 142(2010), 103-118) we have

(9.3)

$$\binom{\frac{2(p-1)}{3}}{\frac{p-1}{3}} \equiv \Big(-L+\frac{p}{L}+\frac{p^2}{L^3}\Big)(1+p^2U_{p-3})$$
$$\equiv -L+\frac{p}{L}+p^2\Big(\frac{1}{L^3}-LU_{p-3}\Big) \pmod{p^3}.$$