

Euler numbers modulo 2^n

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Abstract

Let $\{E_n\}$ be the Euler numbers. In the paper we give a general congruence modulo $2^{(m+2)n}$ involving for E_b and for $E_{2^m k+b}$, where k, m, n are positive integers and $b \in \{0, 2, 4, \dots\}$.

MSC: Primary 11B68, Secondary 11A07

Keywords: Euler number; Congruence

1. Introduction

The Euler numbers $\{E_n\}$ are given by

$$\frac{2e^t}{e^{2t} + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} \quad (|t| < \frac{\pi}{2}).$$

It is well known that

$$E_0 = 1, \quad E_{2n-1} = 0, \quad \sum_{r=0}^n \binom{2n}{2r} E_{2r} = 0 \quad (n = 1, 2, 3, \dots).$$

The first few Euler numbers are shown below:

$$E_0 = 1, \quad E_2 = -1, \quad E_4 = 5, \quad E_6 = -61, \quad E_8 = 1385, \quad E_{10} = -50521, \\ E_{12} = 2702765, \quad E_{14} = -199360981, \quad E_{16} = 19391512145.$$

Let \mathbb{N} be the set of positive integers. For $m \in \mathbb{N}$ and $b \in \{0, 2, 4, \dots\}$, in 1875 Stern (see [1],[6]) showed that

$$(1.1) \quad E_{2^m+b} \equiv 2^m + E_b \pmod{2^{m+1}}.$$

¹The author is supported by the Natural Sciences Foundation of China (grant 10971078).

Let $[x]$ be the greatest integer not exceeding x . In 2005, Z.W.Sun [5] showed that for any $k, n \in \mathbb{N}$,

$$(1.2) \quad \frac{3^{2k+1} + 1}{4} E_{2k} \equiv \frac{3^{2k}}{2} \sum_{j=0}^{2^n-1} (-1)^{j-1} (2j+1)^{2k} \left[\frac{3j+1}{2^n} \right] \pmod{2^n}$$

and deduced (1.1) from (1.2).

For $k, m, n \in \mathbb{N}$ and $b \in \{0, 2, 4, \dots\}$, in [4, Corollary 7.4] the author showed that

$$(1.3) \quad E_{2^m k+b} \equiv \sum_{r=0}^{n-1} (-1)^{n-1-r} \binom{k-1-r}{n-1-r} \binom{k}{r} E_{2^m r+b} \pmod{2^{mn+n-\alpha}},$$

where $\alpha \in \mathbb{N}$ is given by $2^{\alpha-1} \leq n < 2^\alpha$.

In Section 2, we give several basic lemmas. In Section 3, using (1.2) and some results in [2,3,4] we mainly deduce a general congruence modulo $2^{(m+2)n}$ involving for E_b and for $E_{2^m k+b}$, where $b, k \in \{0, 1, 2, \dots\}$, $2 \mid b$ and $n \in \mathbb{N}$, see Theorem 3.2. As consequences, we show that for $m \geq 4$,

$$E_{2^m k+b} \equiv \begin{cases} E_b + 5 \cdot 2^m k \pmod{2^{m+4}} & \text{if } b \equiv 0, 6 \pmod{8}, \\ E_b - 3 \cdot 2^m k \pmod{2^{m+4}} & \text{if } b \equiv 2, 4 \pmod{8} \end{cases}$$

and

$$E_{4k} \equiv \frac{1 + 688k - 384k^2}{1 - 1316k - 1184k^2 - 1536k^3} \pmod{4096}.$$

2. Basic lemmas

Lemma 2.1. *For $n \in \mathbb{N}$ we have*

$$\sum_{j=0}^{2^n-1} (-1)^{j-1} \left[\frac{3j+1}{2^n} \right] = 2.$$

Proof. For $j \in \{0, 1, \dots, 2^n - 1\}$ we have

$$(2.1) \quad \left[\frac{3j+1}{2^n} \right] = \begin{cases} 0 & \text{if } j < \frac{2^n-1}{3}, \\ 1 & \text{if } \frac{2^n-1}{3} \leq j < \frac{2^{n+1}-1}{3}, \\ 2 & \text{if } \frac{2^{n+1}-1}{3} \leq j \leq 2^n - 1. \end{cases}$$

Thus

$$\begin{aligned} & \sum_{j=0}^{2^n-1} (-1)^{j-1} \left[\frac{3j+1}{2^n} \right] \\ &= \begin{cases} \sum_{j=(2^n-1)/3}^{(2^{n+1}-2)/3} (-1)^{j-1} + 2 \sum_{j=(2^{n+1}+1)/3}^{2^n-1} (-1)^{j-1} & \text{if } 2 \mid n, \\ \sum_{j=(2^n+1)/3}^{(2^{n+1}-4)/3} (-1)^{j-1} + 2 \sum_{j=(2^{n+1}-1)/3}^{2^n-1} (-1)^{j-1} & \text{if } 2 \nmid n \end{cases} \\ &= 0 + 2 = 2. \end{aligned}$$

Lemma 2.2. *Let $b \in \{0, 2, 4, \dots\}$ and $n \in \mathbb{N}$. Then*

$$\sum_{k=0}^n \binom{n}{k} (-1)^k (3 + 3^{-2k-b}) E_{2k+b} \equiv 0 \pmod{2^{3n+2}}.$$

Proof. By Lemma 2.1 we know that (1.2) is also true for $k = 0$ and $n \in \mathbb{N}$. Thus, replacing n with $3n$ and k with $k + b/2$ in (1.2) we see that for $k \in \{0, 1, 2, \dots\}$ and $b \in \{0, 2, 4, \dots\}$,

$$(3 + 3^{-2k-b}) E_{2k+b} \equiv 2 \sum_{j=0}^{2^{3n}-1} (-1)^{j-1} (2j+1)^{2k+b} \left[\frac{3j+1}{2^{3n}} \right] \pmod{2^{3n+2}}.$$

Therefore,

$$\begin{aligned} & \sum_{k=0}^n \binom{n}{k} (-1)^k (3 + 3^{-2k-b}) E_{2k+b} \\ & \equiv 2 \sum_{j=0}^{2^{3n}-1} (-1)^{j-1} \left[\frac{3j+1}{2^{3n}} \right] \sum_{k=0}^n \binom{n}{k} (-1)^k (2j+1)^{2k+b} \\ & = 2 \sum_{j=0}^{2^{3n}-1} (-1)^{j-1} \left[\frac{3j+1}{2^{3n}} \right] (1 - (2j+1)^2)^n (2j+1)^b \\ & = 2^{3n+1} \sum_{j=0}^{2^{3n}-1} (-1)^{j-1} \left[\frac{3j+1}{2^{3n}} \right] \left(\frac{1 - (2j+1)^2}{8} \right)^n (2j+1)^b \pmod{2^{3n+2}}. \end{aligned}$$

Since

$$\begin{aligned} \left(\frac{1 - (2j+1)^2}{8} \right)^n & \equiv \frac{1 - (2j+1)^2}{8} = -\frac{j(j+1)}{2} \\ & \equiv \begin{cases} 1 \pmod{2} & \text{if } j \equiv 1, 2 \pmod{4}, \\ 0 \pmod{2} & \text{if } j \equiv 0, 3 \pmod{4}, \end{cases} \end{aligned}$$

using (2.1) and the fact $\frac{2^{3n}-1}{3} \equiv \frac{2^{3n+1}-1}{3} \equiv 3 \pmod{4}$ we see that

$$\begin{aligned} & \sum_{j=0}^{2^{3n}-1} (-1)^{j-1} \left[\frac{3j+1}{2^{3n}} \right] \left(\frac{1 - (2j+1)^2}{8} \right)^n (2j+1)^b \\ & \equiv \sum_{\substack{\frac{2^{3n}-1}{3} \leq j < \frac{2^{3n+1}-1}{3} \\ j \equiv 1, 2 \pmod{4}}} (-1)^{j-1} = 0 \pmod{2}. \end{aligned}$$

Now combining all the above we obtain the result.

Lemma 2.3. *Let $b \in \{0, 2, 4, \dots\}$ and $n \in \mathbb{N}$. Then*

$$\sum_{k=0}^n \binom{n}{k} (-1)^k \frac{3^{2k+b+1} + 1}{4} E_{2k+b} \equiv 0 \pmod{2^{3n}}.$$

Proof. Lemma 2.1 of [3] states that for any functions f and g ,

$$\begin{aligned} & \sum_{k=0}^n \binom{n}{k} (-1)^k f(k) g(k) \\ &= \sum_{s=0}^n \binom{n}{s} \left(\sum_{r=0}^s \binom{s}{r} (-1)^r \sum_{j=0}^{n-s+r} \binom{n-s+r}{j} (-1)^j f(j) \right) \sum_{i=0}^s \binom{s}{i} (-1)^i g(i). \end{aligned}$$

Set $f(k) = \frac{3+3^{-2k-b}}{4} E_{2k+b}$ and $g(k) = 3^{2k+b}$. Then

$$\sum_{i=0}^s \binom{s}{i} (-1)^i g(i) = \sum_{i=0}^s \binom{s}{i} (-1)^i 3^{2i+b} = (1-9)^s 3^b \equiv 0 \pmod{2^{3s}}$$

and

$$\sum_{j=0}^{n-s+r} \binom{n-s+r}{j} (-1)^j f(j) \equiv 0 \pmod{2^{3(n-s+r)}}$$

by Lemma 2.2. As $2^{3(n-s+r)} \cdot 2^{3s} = 2^{3n+3r} \equiv 0 \pmod{2^{3n}}$, from the above we see that

$$\begin{aligned} & \sum_{k=0}^n \binom{n}{k} (-1)^k \frac{3^{2k+b+1} + 1}{4} E_{2k+b} \\ &= \sum_{k=0}^n \binom{n}{k} (-1)^k f(k) g(k) \\ &= \sum_{s=0}^n \binom{n}{s} \left(\sum_{r=0}^s \binom{s}{r} (-1)^r \sum_{j=0}^{n-s+r} \binom{n-s+r}{j} (-1)^j f(j) \right) \sum_{i=0}^s \binom{s}{i} (-1)^i g(i) \\ &\equiv 0 \pmod{2^{3n}}. \end{aligned}$$

This proves the lemma.

3. Main results

Theorem 3.1. *Let $b, k \in \{0, 1, 2, \dots\}$, $2 \mid b$ and $m, n, t \in \mathbb{N}$. Then*

$$\begin{aligned} & \frac{3^{2^m kt+b+1} + 1}{4} E_{2^m kt+b} \\ &\equiv \sum_{r=0}^{n-1} (-1)^{n-1-r} \binom{k-1-r}{n-1-r} \binom{k}{r} \frac{3^{2^m rt+b+1} + 1}{4} E_{2^m rt+b} \pmod{2^{(m+2)n}}. \end{aligned}$$

Moreover,

$$\begin{aligned} & \frac{3^{2^m kt+b+1} + 1}{4} E_{2^m kt+b} \\ &\equiv \sum_{r=0}^{n-1} (-1)^{n-1-r} \binom{k-1-r}{n-1-r} \binom{k}{r} \frac{3^{2^m rt+b+1} + 1}{4} E_{2^m rt+b} \end{aligned}$$

$$+ 2^{(m+2)n} \binom{k}{n} (-t)^n \frac{\sum_{r=0}^n \binom{n}{r} (-1)^r \frac{3^{2r+b+1}+1}{4} E_{2r+b}}{2^{3n}} \pmod{2^{(m+2)n+2}}.$$

Proof. Let $f(k) = \frac{3^{2k+b+1}+1}{4} E_{2k+b}$ and $A_k = 2^{-k} \sum_{r=0}^k \binom{k}{r} (-1)^r f(r)$. From Lemma 2.3 we know that $2^{2k} \mid A_k$. Putting $p = 2$ and $f(k) = \frac{3^{2k+b+1}+1}{4} E_{2k+b}$ in a formula in [4, p.88] we have

$$\begin{aligned} & \sum_{r=0}^n \binom{n}{r} (-1)^r f(2^{m-1}rt) \\ &= 2^{mn} t^n A_n + \sum_{k=n+1}^{2^{m-1}nt} (-2)^n (-1)^k A_k \\ & \quad \times \sum_{j=n}^k (-1)^{k-j} \frac{s(k,j)j!}{k!} 2^{k-j} \cdot \frac{S(j,n)n!}{j!} 2^{j-n} \cdot (2^{m-1}t)^j, \end{aligned}$$

where $\{s(n,k)\}$ and $\{S(n,k)\}$ are Stirling numbers given by

$$x(x-1)\cdots(x-n+1) = \sum_{k=0}^n (-1)^{n-k} s(n,k) x^k$$

and

$$x^n = \sum_{k=0}^n S(n,k) x(x-1)\cdots(x-k+1).$$

By [4, Lemma 4.2], for $j, k, n \in \mathbb{N}$, $\frac{s(k,j)j!}{k!} 2^{k-j}$ and $\frac{S(j,n)n!}{j!} 2^{j-n}$ are rational 2-integers. Thus, by the above and the fact $2^{2k} \mid A_k$ we have

$$\sum_{r=0}^n \binom{n}{r} (-1)^r f(2^{m-1}rt) \equiv 2^{(m+2)n} t^n \cdot \frac{A_n}{2^{2n}} \pmod{2^{(m+2)n+2}}.$$

From [2, Lemma 2.1] we know that

$$\begin{aligned} f(2^{m-1}kt) &= \sum_{r=0}^{n-1} (-1)^{n-1-r} \binom{k-1-r}{n-1-r} \binom{k}{r} f(2^{m-1}rt) \\ & \quad + \sum_{r=n}^k \binom{k}{r} (-1)^r \sum_{s=0}^r \binom{r}{s} (-1)^s f(2^{m-1}st). \end{aligned}$$

For $r \geq n+1$ we have $(m+2)r \geq (m+2)(n+1) \geq (m+2)n+2$. Thus applying the above we obtain $\sum_{s=0}^r \binom{r}{s} (-1)^s f(2^{m-1}st) \equiv 0 \pmod{2^{(m+2)n+2}}$ and so

$$\begin{aligned} f(2^{m-1}kt) &\equiv \sum_{r=0}^{n-1} (-1)^{n-1-r} \binom{k-1-r}{n-1-r} \binom{k}{r} f(2^{m-1}rt) \\ & \quad + \binom{k}{n} (-t)^n \cdot 2^{(m+2)n} \cdot \frac{A_n}{2^{2n}} \pmod{2^{(m+2)n+2}}. \end{aligned}$$

So the theorem is proved.

Taking $m = t = 1$ and $b = 0$ in Theorem 3.1 we have the following congruence.

Corollary 3.2. *For any nonnegative integer k and positive integer n we have*

$$\frac{3^{2k+1} + 1}{4} E_{2k} \equiv \sum_{r=0}^{n-1} (-1)^{n-1-r} \binom{k-1-r}{n-1-r} \binom{k}{r} \frac{3^{2r+1} + 1}{4} E_{2r} \pmod{2^{3n}}.$$

Theorem 3.3. *Let $b, k \in \{0, 1, 2, \dots\}$, $2 \mid b$ and $m, n, t \in \mathbb{N}$. Then*

$$E_{2^m k t + b} \equiv \frac{\sum_{r=0}^{n-1} (-1)^{n-1-r} \binom{k-1-r}{n-1-r} \binom{k}{r} \frac{3^{2^m r t + b + 1} + 1}{4} E_{2^m r t + b}}{\frac{3^{b+1} + 1}{4} + 3^{b+1} \sum_{r=1}^n \binom{k}{r} \frac{(3^{2^m t} - 1)^r}{4}} \pmod{2^{(m+2)n}}.$$

Proof. As $3^{2^{m+1}} - 1 = (3^{2^m} - 1)(3^{2^m} + 1)$, by induction we see that $2^{m+2} \mid (3^{2^m} - 1)$ and so $2^{m+2} \mid (3^{2^m t} - 1)$. Thus

$$\begin{aligned} \frac{1}{4} (3^{2^m k t + b + 1} + 1) &= \frac{1}{4} (1 + 3^{b+1} (1 + (3^{2^m t} - 1))^k) \\ &= \frac{3^{b+1} + 1}{4} + \frac{3^{b+1}}{4} \sum_{r=1}^k \binom{k}{r} (3^{2^m t} - 1)^r \\ &\equiv \frac{3^{b+1} + 1}{4} + \frac{3^{b+1}}{4} \sum_{r=1}^n \binom{k}{r} (3^{2^m t} - 1)^r \pmod{2^{(m+2)n}}. \end{aligned}$$

Note that $\frac{1}{4} (3^{2^m k t + b + 1} + 1) \equiv \frac{1}{4} (3 + 1) = 1 \pmod{2}$. Applying the above and Theorem 3.1 we obtain the result.

Corollary 3.4. *For any nonnegative integer k and positive integer n we have*

$$E_{2k} \equiv \frac{\sum_{r=0}^{n-1} (-1)^{n-1-r} \binom{k-1-r}{n-1-r} \binom{k}{r} \frac{3^{2r+1} + 1}{4} E_{2r}}{1 + 3 \sum_{r=1}^n \binom{k}{r} 2^{3r-2}} \pmod{2^{3n}}.$$

Proof. Putting $m = t = 1$ and $b = 0$ in Theorem 3.3 we deduce the result.

Taking $n = 1, 2, 3$ in Corollary 3.4 we have:

Corollary 3.5. *For any nonnegative integer k we have*

$$\begin{aligned} E_{2k} &\equiv \frac{1}{1 - 2k} \pmod{8}, \\ E_{2k} &\equiv \frac{1 - 8k}{1 - 18k + 24k^2} \pmod{64}, \\ E_{2k} &\equiv \frac{1 - 168k + 160k^2}{1 + 110k - 168k^2 + 64k^3} \pmod{512}. \end{aligned}$$

We note that different congruences for $E_{2k} \pmod{64}$ and $E_{2k} \pmod{256}$ have been given by the author in [4, p.111].

Corollary 3.6. *For any $k \in \{0, 1, 2, \dots\}$ and $n \in \mathbb{N}$ we have*

$$E_{4k} \equiv \frac{\sum_{r=0}^{n-1} (-1)^{n-1-r} \binom{k-1-r}{n-1-r} \binom{k}{r} \frac{3^{4r+1} + 1}{4} E_{4r}}{1 + 3 \sum_{r=1}^n \binom{k}{r} 4^{2r-1} 5^r} \pmod{2^{4n}}$$

and

$$E_{4k+2} \equiv \frac{\sum_{r=0}^{n-1} (-1)^{n-1-r} \binom{k-1-r}{n-1-r} \binom{k}{r} \frac{3^{4r+3} + 1}{4} E_{4r+2}}{7 + 27 \sum_{r=1}^n \binom{k}{r} 4^{2r-1} 5^r} \pmod{2^{4n}}.$$

Proof. Putting $m = 2$, $t = 1$ and $b = 0, 2$ in Theorem 3.3 we deduce the result.

Taking $n = 3$ in Corollary 3.6 we deduce the following result.

Corollary 3.7. *For any nonnegative integer k we have*

$$E_{4k} \equiv \frac{1 + 688k - 384k^2}{1 - 1316k - 1184k^2 - 1536k^3} \pmod{4096}$$

and

$$E_{4k+2} \equiv \frac{-7 - 1232k + 640k^2}{7 + 444k + 1632k^2 - 1536k^3} \pmod{4096}.$$

Putting $m = 3$ and $t = 1$ in Theorem 3.3 we have:

Corollary 3.8. *Let $k \in \{0, 1, 2, \dots\}$, $b \in \{0, 2, 4, \dots\}$ and $n \in \mathbb{N}$. Then*

$$E_{8k+b} \equiv \frac{\sum_{r=0}^{n-1} (-1)^{n-1-r} \binom{k-1-r}{n-1-r} \binom{k}{r} \frac{3^{8r+b+1}+1}{4} E_{8r+b}}{\frac{3^{b+1}+1}{4} + 3^{b+1} \sum_{r=1}^n \binom{k}{r} 2^{5r-2} 205^r} \pmod{2^{5n}}.$$

Theorem 3.9. *Let $b \in \{0, 2, 4, \dots\}$ and $k, m \in \mathbb{N}$ with $m \geq 4$. Then*

$$E_{2^m k+b} \equiv \begin{cases} E_b + 5 \cdot 2^m k \pmod{2^{m+4}} & \text{if } b \equiv 0, 6 \pmod{8}, \\ E_b - 3 \cdot 2^m k \pmod{2^{m+4}} & \text{if } b \equiv 2, 4 \pmod{8}. \end{cases}$$

We also have

$$E_{8k+b} \equiv \begin{cases} E_b - 24k \pmod{128} & \text{if } b \equiv 0, 6 \pmod{8}, \\ E_b + 40k \pmod{128} & \text{if } b \equiv 2, 4 \pmod{8}. \end{cases}$$

Proof. Suppose $m \geq 3$. Taking $n = t = 1$ in Theorem 3.1 we see that

$$(3.1) \quad \begin{aligned} & \frac{3^{2^m k+b+1} + 1}{4} E_{2^m k+b} \\ & \equiv \frac{3^{b+1} + 1}{4} E_b - 2^{m+2} k \cdot \frac{1}{8} \left(\frac{3^{b+1} + 1}{4} E_b - \frac{3^{b+3} + 1}{4} E_{b+2} \right) \pmod{2^{m+4}}. \end{aligned}$$

For $r \geq 3$ we see that $4^{r-3}/r$ is a 2-integer. Thus,

$$\begin{aligned} 3^{2^m} - 1 &= (1 - 4)^{2^m} - 1 = \sum_{r=1}^{2^m} \binom{2^m}{r} (-4)^r = \sum_{r=1}^{2^m} \frac{2^m}{r} \binom{2^m - 1}{r - 1} (-4)^r \\ &\equiv 2^m \cdot (-4) + 2^{m-1} (2^m - 1) (-4)^2 \equiv -3 \cdot 2^{m+2} \pmod{2^{m+6}}. \end{aligned}$$

Therefore,

$$3^{2^m k} \equiv (1 - 3 \cdot 2^{m+2})^k \equiv 1 - 3k \cdot 2^{m+2} \pmod{2^{m+6}}$$

and so

$$\frac{3^{2^m k+b+1} + 1}{4} \equiv \frac{(1 - 3k \cdot 2^{m+2}) 3^{b+1} + 1}{4} = \frac{3^{b+1} + 1}{4} - 3^{b+2} \cdot 2^m k \pmod{2^{m+4}}.$$

This together with (3.1) yields

$$(3.2) \quad \begin{aligned} & \frac{3^{b+1} + 1}{4} (E_{2^m k + b} - E_b) \\ & \equiv 2^m k \cdot 3^{b+2} E_{2^m k + b} - 2^{m+2} k \cdot \frac{1}{8} \left(\frac{3^{b+1} + 1}{4} E_b - \frac{3^{b+3} + 1}{4} E_{b+2} \right) \pmod{2^{m+4}}. \end{aligned}$$

It is clear that

$$\begin{aligned} \frac{3^{b+1} + 1}{4} &= \frac{3(1+8)^{b/2} + 1}{4} \equiv \frac{3(1 + \frac{b}{2} \cdot 8 + \binom{b/2}{2} \cdot 8^2) + 1}{4} \\ &= 1 + 3b + 24 \frac{b}{2} \left(\frac{b}{2} - 1 \right) \equiv \begin{cases} 3b + 1 \pmod{32} & \text{if } b \equiv 0, 2 \pmod{8}, \\ 3b - 15 \pmod{32} & \text{if } b \equiv 4, 6 \pmod{8} \end{cases} \end{aligned}$$

and so

$$\frac{3^{b+3} + 1}{4} \equiv \begin{cases} 3(b+2) + 1 = 3b + 7 \pmod{32} & \text{if } b \equiv 0, 6 \pmod{8}, \\ 3(b+2) - 15 = 3b - 9 \pmod{32} & \text{if } b \equiv 2, 4 \pmod{8}. \end{cases}$$

From Corollary 3.5 or [4, Corollary 7.7] we know that

$$E_b \equiv \begin{cases} b + 1 \pmod{32} & \text{if } 4 \mid b, \\ b - 3 \pmod{32} & \text{if } 4 \mid b - 2 \end{cases}$$

and so

$$E_{b+2} \equiv \begin{cases} b + 3 \pmod{32} & \text{if } 4 \mid b - 2, \\ b - 1 \pmod{32} & \text{if } 4 \mid b. \end{cases}$$

Thus,

$$(3.3) \quad \begin{aligned} & \frac{3^{b+1} + 1}{4} E_b - \frac{3^{b+3} + 1}{4} E_{b+2} \\ & \equiv \begin{cases} (3b+1)(b+1) - (3b+7)(b-1) = 8 \pmod{32} & \text{if } b \equiv 0 \pmod{8}, \\ (3b+1)(b-3) - (3b-9)(b+3) = 24 - 8b \pmod{32} & \text{if } b \equiv 2 \pmod{8}, \\ (3b-15)(b+1) - (3b-9)(b-1) = -24 \pmod{32} & \text{if } b \equiv 4 \pmod{8}, \\ (3b-15)(b-3) - (3b+7)(b+3) = -40b + 24 \pmod{32} & \text{if } b \equiv 6 \pmod{8} \end{cases} \end{aligned}$$

and therefore

$$\frac{1}{8} \left(\frac{3^{b+1} + 1}{4} E_b - \frac{3^{b+3} + 1}{4} E_{b+2} \right) \equiv 1 \pmod{4}.$$

Substituting this into (3.2) we obtain

$$\frac{3^{b+1} + 1}{4} (E_{2^m k + b} - E_b) \equiv 2^m k \cdot 3^{b+2} E_{2^m k + b} - 2^{m+2} k \pmod{2^{m+4}}.$$

Since $(3^{b+1} + 1)/4$ is odd, we deduce

$$(3.4) \quad E_{2^m k + b} - E_b \equiv \frac{2^m k}{(3^{b+1} + 1)/4} (3^{b+2} E_{2^m k + b} - 4) \pmod{2^{m+4}}.$$

From the previous argument we have $\frac{3^{b+1}+1}{4} \equiv 3b+1 \pmod{16}$ and

$$E_{2^m k+b} \equiv \begin{cases} 2^m k + b + 1 \pmod{16} & \text{if } b \equiv 0 \pmod{4}, \\ 2^m k + b - 3 \pmod{16} & \text{if } b \equiv 2 \pmod{4}. \end{cases}$$

Thus,

$$\frac{1}{(3^{b+1}+1)/4} \equiv \frac{1}{3b+1} \equiv \begin{cases} 1-3b \equiv 1+b \pmod{16} & \text{if } b \equiv 0 \pmod{4}, \\ 13-3b \equiv -3(1+b) \pmod{16} & \text{if } b \equiv 2 \pmod{4}. \end{cases}$$

We also have

$$3^{b+2} \equiv \begin{cases} 9 \cdot 81^{\frac{b}{4}} \equiv 9 \pmod{16} & \text{if } b \equiv 0 \pmod{4}, \\ 81^{\frac{b+2}{4}} \equiv 1 \pmod{16} & \text{if } b \equiv 2 \pmod{4}. \end{cases}$$

Hence, by the above and (3.4) we have

$$E_{2^m k+b} - E_b \equiv \begin{cases} (1+b) \cdot 2^m k(9(2^m k+b+1)-4) \pmod{2^{m+4}} & \text{if } 4 \mid b, \\ -3(1+b) \cdot 2^m k((2^m k+b-3)-4) \pmod{2^{m+4}} & \text{if } 4 \mid b-2. \end{cases}$$

For $m=3$ we have $2^{2m} = 2^{m+3}$. For $m \geq 4$ we have $2^{m+4} \mid 2^{2m}$. Therefore

$$E_{2^m k+b} - E_b \equiv \begin{cases} (1+b) \cdot 2^m k(9b+5) + 2^{m+3} \left[\frac{3}{m} \right] k \pmod{2^{m+4}} & \text{if } 4 \mid b, \\ -3(1+b) \cdot 2^m k(b-7) + 2^{m+3} \left[\frac{3}{m} \right] k \pmod{2^{m+4}} & \text{if } 4 \mid b-2. \end{cases}$$

It is easily seen that

$$(1+b)(9b+5) = 9b^2 + 14b + 5 \equiv 2b + 5 \equiv \begin{cases} 5 \pmod{16} & \text{if } 8 \mid b, \\ -3 \pmod{16} & \text{if } 8 \mid b-4 \end{cases}$$

and

$$-3(1+b)(b-7) = -3(b-2)^2 + 6b + 33 \equiv -2b + 1 \equiv \begin{cases} 5 \pmod{16} & \text{if } 8 \mid b-6, \\ -3 \pmod{16} & \text{if } 8 \mid b-2. \end{cases}$$

So the result follows.

Corollary 3.10. *Let $b \in \{0, 2, 4, \dots\}$ and $k, m \in \mathbb{N}$ with $m \geq 3$. Then*

$$E_{2^m k+b} \equiv E_b + 5 \cdot 2^m k \pmod{2^{m+3}}.$$

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