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# GENERALIZED LEGENDRE POLYNOMIALS AND RELATED SUPERCONGRUENCES

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Abstract. For any positive integer $n$ and variables $a$ and $x$ we define the generalized Legendre polynomial $P_n(a,x)$ by $P_n(a,x)=\sum_{k=0}^n\binom ak\binom{-1-a}k(\frac{1-x}2)^k$. Let $p$ be an odd prime. In this paper we prove many congruences modulo $p^2$ related to $P_{p-1}(a,x)$. For example, we show that $P_{p-1}(a,x)\equiv(-1)^{\langle a\rangle_p}P_{p-1}(a,-x)\pmod{p^2}$, where $a$ is a rational $p-adic$ integer and $\langle a\rangle_p$ is the least nonnegative residue of $a$ modulo $p$. We also generalize some congruences of Zhi-Wei Sun, and establish congruences for $\sum_{k=0}^{p-1}\binom{2k}k\binom{3k}k/54^k$ and $\sum_{k=0}^{p-1}\binom ak\binom{b-a}k\pmod{p^2}$.

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## 1. Introduction.

Let $n$ be a nonnegative integer and let $[\cdot]$ be the greatest integer function. Then the famous Legendre polynomial $P_n(x)$ is given by

$$P_n(x)=\frac1{2^n}\sum_{k=0}^{[n/2]}\binom nk(-1)^k\binom{2n-2k}n x^{n-2k}=\frac1{2^n\cdot n!}\cdot\frac{d^n}{dx^n}(x^2-1)^n, \tag{1.1}$$

see for example [B, pp.179-180]. For any positive integer $n$ and variables $a$ and $x$ we introduce the generalized Legendre polynomial

$$\begin{aligned}P_n(a,x)&=\sum_{k=0}^n\binom ak\binom{-1-a}k\Big(\frac{1-x}2\Big)^k=\sum_{k=0}^n\binom ak\binom{a+k}k\Big(\frac{x-1}2\Big)^k\\&=\sum_{k=0}^n\binom{a+k}{2k}\binom{2k}k\Big(\frac{x-1}2\Big)^k.\end{aligned}\tag{1.2}$$

We note that $\binom{-1-a}k=(-1)^k\binom{a+k}k$ and $\binom ak\binom{a+k}k=\binom{a+k}{2k}\binom{2k}k$. Clearly $P_n(a,x)=P_n(-1-a,x)$ and $P_n(n,x)=P_n(x)$ (see [B, p.180]).

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Let $p>3$ be a prime. In 2003, based on his work concerning hypergeometric functions and Calabi-Yau manifolds, Rodriguez-Villegas [RV] conjectured the following congruences:

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}\equiv\left(\frac{-1}{p}\right)\pmod{p^2}, \tag{1.3}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{27^k}\equiv\left(\frac{-3}{p}\right)\pmod{p^2}, \tag{1.4}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{64^k}\equiv\left(\frac{-2}{p}\right)\pmod{p^2}, \tag{1.5}$$

$$\sum_{k=0}^{p-1}\frac{\binom{3k}{k}\binom{6k}{3k}}{432^k}\equiv\left(\frac{-1}{p}\right)\pmod{p^2}, \tag{1.6}$$

where $(\frac{a}{p})$ is the Legendre symbol. These congruences were later confirmed by Mortenson [M1-M2] via the Gross-Koblitz formula. Recently the author's brother Zhi-Wei Sun [Su1] posed more conjectures concerning the following sums modulo $p^2$:

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}x^k,\ \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{27^k}x^k,\ \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{64^k}x^k,\ \sum_{k=0}^{p-1}\frac{\binom{3k}{k}\binom{6k}{3k}}{432^k}x^k.$$

For the progress on these conjectures see [S2-S5]. As observed by Tauraso [T2], Zudilin and the author ([S2, pp.1916-1917, 1920], [S4, p.1953], [S5, p.182]), we have

$$\binom{-\frac12}{k}^2=\frac{\binom{2k}{k}^2}{16^k},\quad \binom{-\frac13}{k}\binom{-\frac23}{k}=\frac{\binom{2k}{k}\binom{3k}{k}}{27^k},$$
$$\binom{-\frac14}{k}\binom{-\frac34}{k}=\frac{\binom{2k}{k}\binom{4k}{2k}}{64^k},\quad \binom{-\frac16}{k}\binom{-\frac56}{k}=\frac{\binom{3k}{k}\binom{6k}{3k}}{432^k}.$$

This is the motivation that we introduce and study $P_{p-1}(a,x)\pmod{p^2}$.

Let $\mathbb{Z}$ be the set of integers. For a prime $p$ let $\mathbb{Z}_p$ denote the set of rational $p-adic$ integers. For a $p-adic$ integer $a$ let $\langle a\rangle_p\in\{0,1,\dots,p-1\}$ be given by $a\equiv\langle a\rangle_p\pmod p$. Let $p$ be an odd prime and $a\in\mathbb{Z}_p$. In this paper we show that

$$P_{p-1}(a,x)\equiv(-1)^{\langle a\rangle_p}P_{p-1}(a,-x)\pmod{p^2}. \tag{1.7}$$

Note that $P_n(a,1)=1$. Taking $x=-1$ in (1.7) we obtain

$$\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}=P_{p-1}(a,-1)\equiv(-1)^{\langle a\rangle_p}\pmod{p^2}. \tag{1.8}$$

For $a=-\frac12,-\frac13,-\frac14,-\frac16$ we get (1.3)-(1.6) immediately from (1.8). If $\langle a\rangle_p$ is odd, by (1.7) we have

$$\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\frac{1}{2^k}=P_{p-1}(a,0)\equiv 0 \pmod{p^2}. \tag{1.9}$$

This generalizes previous special results in [S2] and [Su4]. If $f(0),f(1),\dots,f(p-1)$ are $p-adic$ integers, we prove the following more general congruence:

$$\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\Big((-1)^{\langle a\rangle_p}f(k)-\sum_{m=0}^{k}\binom{k}{m}(-1)^m f(m)\Big)\equiv 0 \pmod{p^2}. \tag{1.10}$$

(1.7)-(1.9) can be viewed as vast generalizations of some congruences proved in [S2,S4] (with $a=-\frac12$) and [Su3,Su4] (with $a=-\frac13,-\frac14,-\frac16$). When $\langle a\rangle_p\equiv 1 \pmod 2$, taking $f(k)=\binom{2k}{k}/2^{2k}$ in (1.10) we get

$$\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\binom{2k}{k}\frac{1}{4^k}\equiv 0 \pmod{p^2}. \tag{1.11}$$

This implies several conjectures of Rodriguez-Villegas [RV], see (2.7)-(2.10). In this paper we also establish the congruence

$$(a+1)P_{p-1}(a+1,x)-(2a+1)xP_{p-1}(a,x)+aP_{p-1}(a-1,x)\equiv 0 \pmod{p^2}$$

for $a\not\equiv 0,-1 \pmod p$ and use it to prove our main results. As an application, we deduce the congruence for $\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}/54^k \pmod{p^2}$ $(p>3)$, see Theorem 3.4. In Section 4, we obtain a general congruence for $\sum_{k=0}^{p-1}\binom{a}{k}\binom{b-a}{k} \pmod{p^2}$, where $p$ is an odd prime and $a,b\in\mathbb{Z}_p$.

**2. General congruences for $P_{p-1}(a,x) \pmod{p^2}$.**

**Lemma 2.1.** *Let $n$ be a positive integer. Then*

$$\begin{aligned}&(a+1)P_n(a+1,x)-(2a+1)xP_n(a,x)+aP_n(a-1,x)\\&=-2(2a+1)\binom{a}{n}\binom{a+n}{n}\Big(\frac{x-1}{2}\Big)^{n+1}.\end{aligned}$$

Proof. It is clear that

$$a(a+1-k)(a-k)+(a+1)(a+1+k)(a+k)=(2a+1)((a+k)(a-k+1)+2k^2).$$

Thus,

$$\begin{aligned}&a\binom{a-1+k}{2k}+(a+1)\binom{a+1+k}{2k}-(2a+1)\binom{a+k}{2k}-(2a+1)\frac{k}{2k-1}\binom{a+k-1}{2k-2}\\&=\frac{(a+k-1)(a+k-2)\cdots(a-(k-2))}{(2k)!}\big(a(a+1-k)(a-k)\\&\quad+(a+1)(a+1+k)(a+k)-(2a+1)((a+k)(a-k+1)+2k^2)\big)\\&=0.\end{aligned}$$

Therefore,

$$a\binom{a-1+k}{2k}\binom{2k}{k}+(a+1)\binom{a+1+k}{2k}\binom{2k}{k}-(2a+1)\binom{a+k}{2k}\binom{2k}{k}$$
$$=(2a+1)\cdot 2\binom{a+k-1}{2k-2}\binom{2k-2}{k-1}. \tag{2.1}$$

For any negative integer $m$ set $\binom{\alpha}{m}=0$. Using (2.1) we deduce that

$$\begin{aligned}
&(a+1)P_n(a+1,x)-(2a+1)xP_n(a,x)+aP_n(a-1,x)\\
&=\sum_{k=0}^{n}\Big\{(a+1)\binom{a+1+k}{2k}\binom{2k}{k}+a\binom{a-1+k}{2k}\binom{2k}{k}\\
&\qquad-(2a+1)\Big(1+2\cdot\frac{x-1}{2}\Big)\binom{a+k}{2k}\binom{2k}{k}\Big\}\Big(\frac{x-1}{2}\Big)^k\\
&=-(2a+1)\cdot 2\binom{a+n}{2n}\binom{2n}{n}\Big(\frac{x-1}{2}\Big)^{n+1}\\
&\quad+\sum_{k=0}^{n}\Big\{(a+1)\binom{a+1+k}{2k}\binom{2k}{k}+a\binom{a-1+k}{2k}\binom{2k}{k}\\
&\qquad-(2a+1)\Big(\binom{a+k}{2k}\binom{2k}{k}+2\binom{a+k-1}{2k-2}\binom{2k-2}{k-1}\Big)\Big\}\Big(\frac{x-1}{2}\Big)^k\\
&=-2(2a+1)\binom{a}{n}\binom{a+n}{n}\Big(\frac{x-1}{2}\Big)^{n+1}.
\end{aligned}$$

This proves the lemma. □

**Theorem 2.1.** *Let $p$ be an odd prime and $a\in\mathbb{Z}_p$. Then*

$$\begin{aligned}
&(a+1)P_{p-1}(a+1,x)-(2a+1)xP_{p-1}(a,x)+aP_{p-1}(a-1,x)\\
&\equiv\begin{cases}-2p^2\big(\frac{1}{\langle a\rangle_p}+\frac{1}{\langle a\rangle_p+1}\big)\frac{a-\langle a\rangle_p}{p}\big(1+\frac{a-\langle a\rangle_p}{p}\big)\big(\frac{x-1}{2}\big)^p \pmod{p^3} & \text{if } a\not\equiv 0,-1 \pmod p,\\ -2a(p+a+1)\big(\frac{x-1}{2}\big)^p \pmod{p^3} & \text{if } a\equiv 0 \pmod p,\\ 2(a+1)(a-p)\big(\frac{x-1}{2}\big)^p \pmod{p^3} & \text{if } a\equiv -1 \pmod p.\end{cases}
\end{aligned}$$

Proof. Clearly

$$\binom{a}{p-1}\binom{a+p-1}{p-1}=\frac{a(a-1)\cdots(a-(p-2))(a+1)(a+2)\cdots(a+p-1)}{(p-1)!^2}.$$

If $\langle a\rangle_p\neq 0,p-1$, then

$$\begin{aligned}
&a(a-1)\cdots(a-(p-2))(a+1)(a+2)\cdots(a+p-1)\\
&\equiv(a-\langle a\rangle_p)(a+p-\langle a\rangle_p)\{\langle a\rangle_p(\langle a\rangle_p-1)\cdots 2\cdot 1\cdot(p-1)(p-2)\cdots(\langle a\rangle_p+2)\\
&\qquad\times(\langle a\rangle_p+1)(\langle a\rangle_p+2)\cdots(p-1)\cdot 1\cdot 2\cdots(\langle a\rangle_p-1)\}\\
&=p^2\cdot\frac{a-\langle a\rangle_p}{p}\Big(1+\frac{a-\langle a\rangle_p}{p}\Big)\frac{(p-1)!^2}{\langle a\rangle_p(\langle a\rangle_p+1)}\\
&\equiv\frac{p^2}{\langle a\rangle_p(\langle a\rangle_p+1)}\cdot\frac{a-\langle a\rangle_p}{p}\Big(1+\frac{a-\langle a\rangle_p}{p}\Big)\pmod{p^3}.
\end{aligned}$$

If $\langle a\rangle_p = 0$, then

$$\binom{a}{p-1}\binom{a+p-1}{p-1} = a(a+p-1)\frac{(a^2-1^2)(a^2-2^2)\cdots(a^2-(p-2)^2)}{(p-1)!^2}$$
$$\equiv a(a+p-1)\frac{(-1^2)(-2^2)\cdots(-(p-2)^2)}{(p-1)!^2} = -\frac{a(a+p-1)}{(p-1)^2}$$
$$\equiv -a(a+p-1)(p+1)^2 \equiv -a(a+p-1)(2p+1)$$
$$\equiv -a(-2p+a+p-1) = a(p+1-a) \pmod{p^3}.$$

If $\langle a\rangle_p = p-1$, then $-1-a \equiv 0 \pmod p$ and so

$$\binom{a}{p-1}\binom{a+p-1}{p-1} = \binom{-1-a}{p-1}\binom{-1-a+p-1}{p-1}$$
$$\equiv (-1-a)(p+1-(-1-a)) = -(a+1)(p+a+2) \pmod{p^3}.$$

Now putting all the above together with Lemma 2.1 in the case $n = p-1$ we deduce the result. □

**Lemma 2.2.** *Let $p$ be an odd prime and let $t$ and $x$ be $p$-adic integers. Then*

$$P_{p-1}(pt,x) \equiv 1-t+t\Big(\frac{1+x}{2}\Big)^p + t\Big(\frac{1-x}{2}\Big)^p \pmod{p^2} \quad \textit{and so} \quad P_{p-1}(pt,x) \equiv 1 \pmod p.$$

Proof. It is clear that

$$P_{p-1}(pt,x) = \sum_{k=0}^{p-1}\binom{pt}{k}\binom{pt+k}{k}\Big(\frac{x-1}{2}\Big)^k$$
$$= 1 + t\sum_{k=1}^{p-1}\frac{p}{k}\binom{pt-1}{k-1}\binom{pt+k}{k}\Big(\frac{x-1}{2}\Big)^k$$
$$\equiv 1 + t\sum_{k=1}^{p-1}\binom{p}{k}\Big(\frac{x-1}{2}\Big)^k = 1 + t\Big\{\Big(1+\frac{x-1}{2}\Big)^p - 1 - \Big(\frac{x-1}{2}\Big)^p\Big\}$$
$$= 1-t+t\Big\{\Big(\frac{1+x}{2}\Big)^p + \Big(\frac{1-x}{2}\Big)^p\Big\} \pmod{p^2}.$$

Since $\big(\frac{1\pm x}{2}\big)^p \equiv \frac{1\pm x^p}{2} \pmod p$, by the above we obtain $P_{p-1}(pt,x) \equiv 1 \pmod p$. This proves the lemma. □

**Lemma 2.3.** *Let $p$ be an odd prime and let $t$ and $x$ be $p-$adic integers. Then*

$$P_{p-1}(1+pt,x) \equiv (1-t)x + pt\frac{x-1}{2} + t\Big\{(p+1)\frac{x-1}{2}\Big(\Big(\frac{1+x}{2}\Big)^p + \Big(\frac{1-x}{2}\Big)^p\Big)$$
$$+ (2p+1)\Big(\frac{1-x}{2}\Big)^p + \Big(\frac{1+x}{2}\Big)^{p+1} - \Big(\frac{1-x}{2}\Big)^{p+1}\Big\} \pmod{p^2}$$

*and so* $P_{p-1}(1+pt,x)\equiv x \pmod p$.

Proof. It is clear that

$$\begin{aligned}
&P_{p-1}(1+pt,x)\\
&=\sum_{k=0}^{p-1}\binom{pt+1}{k}\binom{pt+1+k}{k}\Big(\frac{x-1}{2}\Big)^k\\
&=1+(pt+1)(pt+2)\frac{x-1}{2}+\sum_{k=2}^{p-1}\frac{pt+1}{k}\cdot\frac{pt}{k-1}\binom{pt-1}{k-2}\binom{pt+1+k}{k}\Big(\frac{x-1}{2}\Big)^k\\
&\equiv 1+(3pt+2)\frac{x-1}{2}+t\sum_{k=2}^{p-1}\binom{p+1}{k}\binom{k+1}{k}\Big(\frac{x-1}{2}\Big)^k\\
&=1+(pt+2-2t)\frac{x-1}{2}+t\sum_{k=1}^{p-1}\binom{p+1}{k}(k+1)\Big(\frac{x-1}{2}\Big)^k \pmod{p^2}
\end{aligned}$$

and so

$$\begin{aligned}
&P_{p-1}(1+pt,x)\\
&\equiv 1+(pt+2-2t)\frac{x-1}{2}\\
&\quad+t\sum_{k=1}^{p-1}\binom{p+1}{k}\Big(\frac{x-1}{2}\Big)^k+t(p+1)\frac{x-1}{2}\sum_{k=1}^{p-1}\binom{p}{k-1}\Big(\frac{x-1}{2}\Big)^{k-1}\\
&=1+(pt+2-2t)\frac{x-1}{2}+t\Big\{\Big(1+\frac{x-1}{2}\Big)^{p+1}-1-(p+1)\Big(\frac{x-1}{2}\Big)^p-\Big(\frac{x-1}{2}\Big)^{p+1}\\
&\quad+(p+1)\frac{x-1}{2}\Big(\Big(1+\frac{x-1}{2}\Big)^p-\Big(\frac{x-1}{2}\Big)^p-p\Big(\frac{x-1}{2}\Big)^{p-1}\Big)\Big\}\\
&=(1-t)x+pt\frac{x-1}{2}+t\Big\{(p+1)\frac{x-1}{2}\Big(\Big(\frac{1+x}{2}\Big)^p+\Big(\frac{1-x}{2}\Big)^p\Big)\\
&\quad+(p+1)^2\Big(\frac{1-x}{2}\Big)^p+\Big(\frac{1+x}{2}\Big)^{p+1}-\Big(\frac{1-x}{2}\Big)^{p+1}\Big\} \pmod{p^2}.
\end{aligned}$$

Observe that $\left(\frac{1\pm x}{2}\right)^p\equiv\frac{1\pm x^p}{2} \pmod p$. From the above we see that

$$\begin{aligned}
P_{p-1}(1+pt,x)&\equiv(1-t)x+t\Big\{\frac{x-1}{2}+\frac{1-x^p}{2}+\frac{1+x}{2}\cdot\frac{1+x^p}{2}-\frac{1-x}{2}\cdot\frac{1-x^p}{2}\Big\}\\
&=x-tx+tx=x \pmod p.
\end{aligned}$$

This completes the proof. □

**Theorem 2.2.** *Let* $p$ *be an odd prime and* $a\in\mathbb{Z}_p$*. Then*

$$P_{p-1}(a,x)\equiv(-1)^{\langle a\rangle_p}P_{p-1}(a,-x) \pmod{p^2}$$

*and so*

$$\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}(x^k-(-1)^{\langle a\rangle_p}(1-x)^k)\equiv 0 \pmod{p^2}.$$

Proof. Suppose $m\in\{1,2,\dots,p-2\}$ and $t\in\mathbb{Z}_p$. From Theorem 2.1 we have

$$\begin{aligned}(m+1+pt)P_{p-1}(m+1+pt,\pm x)-(2(m+pt)+1)(\pm x)P_{p-1}(m+pt,\pm x)\\ +(m+pt)P_{p-1}(m-1+pt,\pm x)\equiv 0 \pmod{p^2}.\end{aligned}$$

Thus,

$$\begin{aligned}&(m+1+pt)(P_{p-1}(m+1+pt,x)-(-1)^{m+1}P_{p-1}(m+1+pt,-x))\\ &\equiv(2(m+pt)+1)x(P_{p-1}(m+pt,x)-(-1)^mP_{p-1}(m+pt,-x))\\ &\quad-(m+pt)(P_{p-1}(m-1+pt,x)-(-1)^{m-1}P_{p-1}(m-1+pt,-x)) \pmod{p^2}.\end{aligned}\tag{2.2}$$

From Lemma 2.2 we know that

$$P_{p-1}(pt,x)\equiv P_{p-1}(pt,-x) \pmod{p^2}.$$

From Lemma 2.3 we see that

$$\begin{aligned}&P_{p-1}(1+pt,x)+P_{p-1}(1+pt,-x)\\ &\equiv pt\Big(\frac{x-1}{2}+\frac{-x-1}{2}\Big)+t\Big\{(p+1)\Big(\frac{x-1}{2}+\frac{-x-1}{2}\Big)\Big(\Big(\frac{1+x}{2}\Big)^p+(\frac{1-x}{2})^p\Big)\\ &\quad+(1+2p)\Big(\Big(\frac{1+x}{2}\Big)^p+(\frac{1-x}{2})^p\Big)\Big\}\\ &\equiv -pt+pt\Big(\Big(\frac{1+x}{2}\Big)^p+(\frac{1-x}{2})^p\Big)\equiv -pt+pt\Big(\frac{1+x^p}{2}+\frac{1-x^p}{2}\Big)\\ &=0 \pmod{p^2}.\end{aligned}$$

Thus,

$$P_{p-1}(m+pt,x)-(-1)^mP_{p-1}(m+pt,-x)\equiv 0 \pmod{p^2}\quad\text{for}\quad m=0,1.$$

By (2.2) and induction we deduce that $P_{p-1}(m+pt,x)-(-1)^mP_{p-1}(m+pt,-x)\equiv 0 \pmod{p^2}$ for all $m=0,1,\dots,p-1$. Since $a=\langle a\rangle_p+pt$ for some $t\in\mathbb{Z}_p$, we see that $P_{p-1}(a,x)\equiv(-1)^{\langle a\rangle_p}P_{p-1}(a,-x) \pmod{p^2}$ and hence

$$\begin{aligned}\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}x^k&=P_{p-1}(a,1-2x)\equiv(-1)^{\langle a\rangle_p}P_{p-1}(a,2x-1)\\ &=(-1)^{\langle a\rangle_p}\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}(1-x)^k \pmod{p^2}.\end{aligned}$$

This completes the proof. □

**Remark 2.1** In the case $a=-\frac12$, Theorem 2.2 was given by the author in [S2] and independently by Tauraso in [T1]. In the cases $a=-\frac13,-\frac14,-\frac16$, Theorem 2.2 was given by Z. W. Sun in [Su4].

**Corollary 2.1.** *Let $p$ be an odd prime and $a \in \mathbb{Z}_p$. Then*

$$\sum_{k=0}^{p-1} \binom{a}{k}\binom{-1-a}{k} \equiv (-1)^{\langle a\rangle_p} \pmod{p^2}.$$

Proof. Taking $x = 1$ in Theorem 2.2 we obtain the result. □

As mentioned in Section 1, taking $a = -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{6}$ in Corollary 2.1 we deduce (1.3)-(1.6).

**Corollary 2.2.** *Let $p$ be an odd prime and $a \in \mathbb{Z}_p$ with $\langle a\rangle_p \equiv 1 \pmod 2$. Then*

$$\sum_{k=0}^{p-1} \binom{a}{k}\binom{-1-a}{k}\frac{1}{2^k} \equiv 0 \pmod{p^2}.$$

Proof. Taking $x = \frac{1}{2}$ in Theorem 2.2 we obtain the result. □

Putting $a = -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{6}$ in Corollary 2.2 we deduce the following congruences:

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{32^k} \equiv 0 \pmod{p^2} \quad \text{for} \quad p \equiv 3 \pmod 4, \tag{2.3}$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}}{54^k} \equiv 0 \pmod{p^2} \quad \text{for} \quad p \equiv 2 \pmod 3, \tag{2.4}$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{4k}{2k}}{128^k} \equiv 0 \pmod{p^2} \quad \text{for} \quad p \equiv 5, 7 \pmod 8, \tag{2.5}$$

$$\sum_{k=0}^{p-1} \frac{\binom{3k}{k}\binom{6k}{3k}}{864^k} \equiv 0 \pmod{p^2} \quad \text{for} \quad p \equiv 3 \pmod 4, \tag{2.6}$$

where $p$ is a prime greater than 3. We remark that (2.3) was conjectured by Z. W. Sun and proved by the author in [S2] and Tauraso in [T1], and (2.4) was conjectured by the author in [S2] and proved by Z. W. Sun in [Su4]. (2.5) and (2.6) were conjectured by Z. W. Sun and finally proved by him in [Su4], although the author proved the corresponding congruences modulo $p$ earlier.

**Lemma 2.4.** *Let $n$ be a positive integer. Then*

$$\sum_{k=0}^{n} \frac{\binom{a}{k}\binom{-1-a}{k}}{k+1} = \frac{\binom{a-1}{n}\binom{-2-a}{n}}{n+1}.$$

Proof. Set

$$f(n) = \frac{\binom{a}{n}\binom{-1-a}{n}}{n+1} \quad \text{and} \quad g(n) = \frac{\binom{a-1}{n}\binom{-2-a}{n}}{n+1}.$$

It is easily seen that $g(n) - g(n-1) = f(n)$. Thus,

$$\sum_{k=0}^{n} f(k) = f(0) + \sum_{k=1}^{n}(g(k) - g(k-1)) = f(0) - g(0) + g(n) = g(n).$$

This proves the lemma. □

**Theorem 2.3.** *Let $p$ be an odd prime and $a \in \mathbb{Z}_p$.*

(i) *If $m \in \{1, 2, \dots, p-1\}$, then*

$$\sum_{k=m}^{p-1} \binom{a}{k}\binom{-1-a}{k}\binom{k}{m}(x^{k-m} - (-1)^{m+\langle a\rangle_p}(1-x)^{k-m}) \equiv 0 \pmod{p^2}.$$

(ii) *If $a \not\equiv 0, -1 \pmod{p}$, then*

$$\sum_{k=0}^{p-2} \binom{a}{k}\binom{-1-a}{k}\frac{x^{k+1} + (-1)^{\langle a\rangle_p}(1-x)^{k+1}}{k+1} \equiv 0 \pmod{p^2}.$$

Proof. By Theorem 2.2,

$$\sum_{k=0}^{p-1} \binom{a}{k}\binom{-1-a}{k}(x^k - (-1)^{\langle a\rangle_p}(1-x)^k) = p^2 f(x),$$

where $f(x)$ is a polynomial of $x$ with rational $p$-integral coefficients and degree at most $p-1$. Since $\frac{d^m}{dx^m}x^k = m!\binom{k}{m}x^{k-m}$ and $\frac{d^m}{dx^m}(1-x)^k = (-1)^k m!\binom{k}{m}(x-1)^{k-m} = (-1)^m m!\binom{k}{m}(1-x)^{k-m}$ for $k \ge m$, we see that

$$\sum_{k=m}^{p-1} \binom{a}{k}\binom{-1-a}{k}\binom{k}{m}(x^{k-m} - (-1)^{m+\langle a\rangle_p}(1-x)^{k-m}) = \frac{p^2}{m!}\cdot\frac{d^m}{dx^m}f(x).$$

As $p \nmid m!$ for $1 \le m \le p-1$ and $\frac{d^m}{dx^m}f(x)$ is a polynomial of $x$ with rational $p$-integral coefficients and degree at most $p-1$, we deduce the first part.

Now we suppose $a \not\equiv 0, -1 \pmod{p}$. It is easy to see that

$$\binom{a}{p-1} \equiv \binom{-1-a}{p-1} \equiv 0 \pmod{p} \quad \text{and so} \quad \binom{a}{p-1}\binom{-1-a}{p-1} \equiv 0 \pmod{p^2}.$$

Thus, by Theorem 2.2 we have

$$\sum_{k=0}^{p-2} \binom{a}{k}\binom{-1-a}{k}(x^k - (-1)^{\langle a\rangle_p}(1-x)^k) = p^2 g(x),$$

where $g(x)$ is a polynomial of $x$ with rational $p$-integral coefficients and degree at most $p-2$. It is easy to see that

$$p^2\int_0^x g(t)dt = \sum_{k=0}^{p-2} \binom{a}{k}\binom{-1-a}{k}\frac{x^{k+1} + (-1)^{\langle a\rangle_p}((1-x)^{k+1} - 1)}{k+1}.$$

Thus, using Lemma 2.4 we get

$$\begin{aligned}&\sum_{k=0}^{p-2}\binom{a}{k}\binom{-1-a}{k}\frac{x^{k+1}+(-1)^{\langle a\rangle_p}(1-x)^{k+1}}{k+1}\\&\equiv(-1)^{\langle a\rangle_p}\sum_{k=0}^{p-2}\frac{\binom ak\binom{-1-a}k}{k+1}=\frac{(-1)^{\langle a\rangle_p}}{p-1}\binom{a-1}{p-2}\binom{-2-a}{p-2}\pmod{p^2}.\end{aligned}$$

Since $a\not\equiv 0,-1 \pmod p$, we see that

$$\binom{a-1}{p-2}\binom{-2-a}{p-2}=\frac{p-1}{a}\binom{a}{p-1}\cdot\frac{p-1}{-1-a}\binom{-1-a}{p-1}\equiv 0 \pmod{p^2}.$$

Thus the second part follows and the proof is complete. □

**Corollary 2.3.** *Let $p$ be an odd prime, $a\in\mathbb{Z}_p$ and $m\in\{1,2,\dots,p-1\}$. Then*

$$\sum_{k=m}^{p-1}\binom{a}{k}\binom{-1-a}{k}\binom{k}{m}\equiv(-1)^{m+\langle a\rangle_p}\binom{a}{m}\binom{-1-a}{m}\pmod{p^2}.$$

Proof. Taking $x=1$ in Theorem 2.3(i) we obtain the result. □

**Corollary 2.4.** *Let $p$ be an odd prime and $a\in\mathbb{Z}_p$ with $\langle a\rangle_p\in\{2,4,\dots,p-3\}$. Then*

$$\sum_{k=0}^{p-2}\binom{a}{k}\binom{-1-a}{k}\frac{1}{2^k(k+1)}\equiv 0\pmod{p^2}.$$

Proof. Taking $x=\frac12$ in Theorem 2.3(ii) we obtain the result. □

**Theorem 2.4.** *Let $p$ be an odd prime and $a\in\mathbb{Z}_p$. If $f(0),f(1),\dots,f(p-1)$ are $p-adic$ integers, then*

$$\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\Big((-1)^{\langle a\rangle_p}f(k)-\sum_{m=0}^{k}\binom{k}{m}(-1)^m f(m)\Big)\equiv 0\pmod{p^2}.$$

Proof. From Corollaries 2.1 and 2.3 we see that

$$\begin{aligned}&\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\sum_{m=0}^{k}\binom{k}{m}(-1)^m f(m)\\&=\sum_{m=0}^{p-1}(-1)^m f(m)\sum_{k=m}^{p-1}\binom{a}{k}\binom{-1-a}{k}\binom{k}{m}\\&\equiv\sum_{m=0}^{p-1}(-1)^m f(m)\cdot(-1)^{m+\langle a\rangle_p}\binom{a}{m}\binom{-1-a}{m}\\&=(-1)^{\langle a\rangle_p}\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}f(k)\pmod{p^2}.\end{aligned}$$

This yields the result. □

**Remark 2.2** In the case $a=-\frac12$, Theorem 2.4 was obtained by the author in 2010. In the cases $a=-\frac13,-\frac14,-\frac16$, Theorem 2.4 was recently obtained by Z.W. Sun in [Su4].

**Theorem 2.5.** *Let $p$ be an odd prime and $a\in\mathbb{Z}_p$ with $\langle a\rangle_p\equiv 1 \pmod 2$. Then*

$$\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\binom{2k}{k}\frac{1}{2^{2k}}\equiv 0 \pmod{p^2}.$$

Proof. Set $f(k)=\binom{2k}{k}/2^{2k}$. From [S1, Example 10] we know that $\sum_{m=0}^{k}\binom{k}{m}(-1)^m f(m) = f(k)$. Thus, applying Theorem 2.4 we deduce the result. □

Putting $a=-\frac12,-\frac13,-\frac14,-\frac16$ in Theorem 2.5 we deduce that for any prime $p>3$,

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}\equiv 0 \pmod{p^2}\quad\text{for}\quad p\equiv 3 \pmod 4, \tag{2.7}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k}\equiv 0 \pmod{p^2}\quad\text{for}\quad p\equiv 5 \pmod 6, \tag{2.8}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k}\equiv 0 \pmod{p^2}\quad\text{for}\quad p\equiv 5,7 \pmod 8, \tag{2.9}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{1728^k}\equiv 0 \pmod{p^2}\quad\text{for}\quad p\equiv 3 \pmod 4. \tag{2.10}$$

Here (2.7) was conjectured by Beukers [Beu] in 1987 and proved by van Hamme [vH]. (2.8)-(2.10) were conjectured by Rodriguez-Villegas [RV] and proved by Z. W. Sun [Su3].

**3. Congruences for $P_{p-1}(a,0) \pmod{p^2}$.**

For given positive integer $n$ and prime $p$ we define

$$H_0=0,\quad H_n=1+\frac12+\frac13+\cdots+\frac1n\quad\text{and}\quad q_p(a)=\frac{a^{p-1}-1}{p}.$$

**Theorem 3.1.** *Let $p$ be an odd prime and $t\in\mathbb{Z}_p$.*

(i) *If $n\in\{0,1,\ldots,\frac{p-3}{2}\}$, then $P_{p-1}(2n+1+pt,0)\equiv 0 \pmod{p^2}$.*

(ii) *If $n\in\{0,1,\ldots,\frac{p-1}{2}\}$, then*

$$P_{p-1}(2n+pt,0)\equiv\binom{\frac{p-1}{2}}{n}\Big(1+p\big((1+t)H_{2n}-\frac{2t+1}{2}H_n-tq_p(2)\big)\Big) \pmod{p^2}.$$

Proof. Putting $x=0$ in Theorem 2.1 we see that

$$P_{p-1}(a+1,0)\equiv-\frac{a}{a+1}P_{p-1}(a-1,0) \pmod{p^2}\quad\text{for}\quad a\not\equiv 0,-1 \pmod p.$$

Assume $n \in \{1, 2, \dots, \frac{p-3}{2}\}$. Then

$$\begin{aligned} &P_{p-1}(2n+1+pt, 0) \\ &\equiv -\frac{2n+pt}{2n+1+pt} P_{p-1}(2n-1+pt, 0) \equiv \cdots \\ &\equiv (-1)^n \frac{2n+pt}{2n+1+pt} \cdot \frac{2n-2+pt}{2n-1+pt} \cdots \frac{2+pt}{3+pt} P_{p-1}(1+pt, 0) \pmod{p^2}. \end{aligned}$$

By Lemma 2.3,

$$P_{p-1}(1+pt, 0) \equiv -\frac{p}{2}t + t\Big(-\frac{p+1}{2} \cdot \frac{1}{2^{p-1}} + (2p+1) \cdot \frac{1}{2^p}\Big) = -\frac{p}{2}t + \frac{p}{2^p}t \equiv 0 \pmod{p^2}.$$

Thus, from the above we deduce that $P_{p-1}(2n+1+pt, 0) \equiv 0 \pmod{p^2}$ for $n = 0, 1, \dots, \frac{p-3}{2}$. This proves (i).

Now let us consider (ii). Assume $n \in \{1, 2, \dots, \frac{p-1}{2}\}$. Then

$$\begin{aligned} P_{p-1}(2n+pt, 0) &\equiv -\frac{2n-1+pt}{2n+pt} P_{p-1}(2n-2+pt, 0) \equiv \cdots \\ &\equiv (-1)^n \frac{2n-1+pt}{2n+pt} \cdot \frac{2n-3+pt}{2n-2+pt} \cdots \frac{3+pt}{4+pt} \cdot \frac{1+pt}{2+pt} P_{p-1}(pt, 0) \\ &\equiv (-1)^n \frac{1 \cdot 3 \cdots (2n-1)(1+pt\sum_{k=1}^n \frac{1}{2k-1})}{2 \cdot 4 \cdots (2n)(1+pt\sum_{k=1}^n \frac{1}{2k})} P_{p-1}(pt, 0) \\ &\equiv \frac{1}{(-4)^n}\binom{2n}{n}\Big(1+pt\sum_{k=1}^n \frac{1}{2k-1}\Big)\Big(1-pt\sum_{k=1}^n \frac{1}{2k}\Big) P_{p-1}(pt, 0) \\ &\equiv \frac{1}{(-4)^n}\binom{2n}{n}(1+pt(H_{2n}-H_n)) P_{p-1}(pt, 0) \pmod{p^2}. \end{aligned}$$

By [S2, Lemma 2.4],

$$\text{(3.1)} \quad \frac{1}{(-4)^n}\binom{2n}{n} \equiv \binom{\frac{p-1}{2}}{n}\Big(1+p\sum_{k=1}^n \frac{1}{2k-1}\Big) = \binom{\frac{p-1}{2}}{n}\Big(1+p(H_{2n}-\frac{1}{2}H_n)\Big) \pmod{p^2}.$$

From Lemma 2.2 we have

$$P_{p-1}(pt, 0) \equiv 1 - t + \frac{t}{2^{p-1}} \equiv 1 - t + t(1-pq_p(2)) = 1 - ptq_p(2) \pmod{p^2}.$$

Therefore, for $n = 0, 1, \dots, \frac{p-1}{2}$ we have

$$\begin{aligned} P_{p-1}(2n+pt, 0) &\equiv \binom{\frac{p-1}{2}}{n}\Big(1+p\big(H_{2n}-\frac{H_n}{2}\big)\Big)\Big(1+pt(H_{2n}-H_n))(1-ptq_p(2)\Big) \\ &\equiv \binom{\frac{p-1}{2}}{n}\Big(1+p\big((1+t)H_{2n}-\frac{2t+1}{2}H_n - tq_p(2)\big)\Big) \pmod{p^2}. \end{aligned}$$

This proves (ii) and hence the proof is complete. □

**Corollary 3.1.** *Let $p$ be an odd prime and let $a$ be a $p$-adic integer with $a \not\equiv 0 \pmod p$. Then $P_{p-1}(a,0) \equiv 0 \pmod{p^2}$ or $P_{p-1}(-a,0) \equiv 0 \pmod{p^2}$.*

Proof. If $a \equiv 2n+1 \pmod p$ with $n \in \{0,1,\ldots,\frac{p-3}{2}\}$, by Theorem 3.1(i) we have $P_{p-1}(a,0) \equiv 0 \pmod{p^2}$. If $a \equiv 2n \pmod p$ for some $n \in \{1,2,\ldots,\frac{p-1}{2}\}$, then $-a \equiv p-2n = 1+2(\frac{p-1}{2}-n) \pmod p$ and so $P_{p-1}(-a,0) \equiv 0 \pmod{p^2}$ by Theorem 3.1(i). $\square$

For $a,b \in \mathbb{Z}$ (not both zero) let $(a,b)$ be the greatest common divisor of $a$ and $b$.

**Theorem 3.2.** *Let $p$ be an odd prime, $m \in \mathbb{Z}$, $m > 1$, $r \in \{\pm 1, \pm 2, \ldots, \pm(m-1)\}$, $(p,m) = (r,m) = 1$ and $p \geq 2m+r$.*

(i) *If $2 \nmid rm$, then*

$$P_{p-1}\Big(\frac{r}{m},0\Big) = \sum_{k=0}^{p-1}\binom{r/m}{k}\binom{-1-r/m}{k}\frac{1}{2^k}$$
$$\equiv \begin{cases} \binom{(p-1)/2}{n}\big(1+p((1-\frac{2s}{m})H_{2n}+(\frac{2s}{m}-\frac{1}{2})H_n+\frac{2s}{m}q_p(2))\big) \pmod{p^2} \\ \qquad \text{if } r \equiv 2sp \pmod m \text{ for some } s \in \{1,2,\ldots,\frac{m-1}{2}\} \\ \qquad \text{and } n = \frac{sp-(m+r)/2}{m}, \\ 0 \pmod{p^2} \quad \text{if } r \equiv (2s-1)p \pmod m \text{ for some } s \in \{1,2,\ldots,\frac{m-1}{2}\}. \end{cases}$$

(ii) *If $2 \nmid m$ and $2 \mid r$, then*

$$P_{p-1}\Big(\frac{r}{m},0\Big) = \sum_{k=0}^{p-1}\binom{r/m}{k}\binom{-1-r/m}{k}\frac{1}{2^k}$$
$$\equiv \begin{cases} \binom{(p-1)/2}{n}\big(1+p((1-\frac{2s}{m})H_{2n}+(\frac{2s}{m}-\frac{1}{2})H_n+\frac{2s}{m}q_p(2))\big) \pmod{p^2} \\ \qquad \text{if } \frac{r}{2} \equiv -sp \pmod m \text{ for some } s \in \{1,2,\ldots,\frac{m-1}{2}\} \\ \qquad \text{and } n = \frac{sp+r/2}{m}, \\ 0 \pmod{p^2} \quad \text{if } \frac{r}{2} \equiv sp \pmod m \text{ for some } s \in \{1,2,\ldots,\frac{m-1}{2}\}. \end{cases}$$

(iii) *If $2 \mid m$ and $2 \nmid r$, then*

$$P_{p-1}\Big(\frac{r}{m},0\Big) = \sum_{k=0}^{p-1}\binom{r/m}{k}\binom{-1-r/m}{k}\frac{1}{2^k}$$
$$\equiv \begin{cases} \binom{(p-1)/2}{n}\big(1+p((1-\frac{s}{m})H_{2n}+(\frac{s}{m}-\frac{1}{2})H_n+\frac{s}{m}q_p(2))\big) \pmod{p^2} \\ \qquad \text{if } r \equiv -sp \pmod{2m} \text{ for some } s \in \{1,2,\ldots,m-1\} \\ \qquad \text{and } n = \frac{sp+r}{2m}, \\ 0 \pmod{p^2} \quad \text{if } r \equiv sp \pmod{2m} \text{ for some } s \in \{1,2,\ldots,m-1\}. \end{cases}$$

Proof. We first consider (i). If $2 \nmid rm$ and $r \equiv (2s-1)p \pmod m$ for some $s \in \{1,2,\ldots,\frac{m-1}{2}\}$, setting $n = \frac{(2s-1)p-r}{2m} - 1$ and $t = -\frac{2s-1}{m}$ we find $n \in \{0,1,\ldots,\frac{p-3}{2}\}$ and

$2n+1+pt=-1-\frac{r}{m}$. Thus, from Theorem 3.1(i) we obtain $P_{p-1}(\frac{r}{m},0)=P_{p-1}(-1-\frac{r}{m},0)=P_{p-1}(2n+1+pt,0)\equiv 0 \pmod{p^2}$. If $2\nmid rm$ and $r\equiv 2sp \pmod m$ with $s\in\{1,2,\ldots,\frac{m-1}{2}\}$, setting $n=\frac{sp-(m+r)/2}{m}$ and $t=-\frac{2s}{m}$ we find $n\in\{1,2,\ldots,\frac{p-1}{2}\}$ and $2n+pt=-1-\frac{r}{m}$. Thus, from Theorem 3.1(ii) we deduce that

$$\begin{aligned}P_{p-1}\Big(\frac{r}{m},0\Big)&=P_{p-1}\Big(-1-\frac{r}{m},0\Big)=P_{p-1}(2n+pt,0)\\&\equiv\binom{(p-1)/2}{n}\Big(1+p\big((1-\frac{2s}{m})H_{2n}+(\frac{2s}{m}-\frac{1}{2})H_n+\frac{2s}{m}q_p(2)\big)\Big) \pmod{p^2}.\end{aligned}$$

This proves (i).

Now we consider (ii). Suppose $2\nmid m$ and $2\mid r$. If $\frac{r}{2}\equiv sp \pmod m$ for some $s\in\{1,2,\ldots,\frac{m-1}{2}\}$, setting $n=\frac{sp-r/2}{m}-1$ and $t=-\frac{2s}{m}$ we find that $n\in\{0,1,\ldots,\frac{p-3}{2}\}$ and $2n+1+pt=-1-\frac{r}{m}$. Thus, from Theorem 3.1(i) we obtain $P_{p-1}(\frac{r}{m},0)=P_{p-1}(-1-\frac{r}{m},0)=P_{p-1}(2n+1+pt,0)\equiv 0 \pmod{p^2}$. If $\frac{r}{2}\equiv -sp \pmod m$ for some $s\in\{1,2,\ldots,\frac{m-1}{2}\}$, setting $n=\frac{sp+r/2}{m}$ and $t=-\frac{2s}{m}$ we find that $n\in\{0,1,\ldots,\frac{p-1}{2}\}$ and $2n+pt=\frac{r}{m}$. Thus, from Theorem 3.1(ii) we deduce the result.

Let us consider (iii). Assume $2\mid m$ and $2\nmid r$. If $r\equiv sp \pmod{2m}$ for some $s\in\{1,2,\ldots,m-1\}$, setting $n=\frac{sp-r}{2m}-1$ and $t=-\frac{s}{m}$ we find that $n\in\{0,1,\ldots,\frac{p-3}{2}\}$ and $2n+1+pt=-1-\frac{r}{m}$. Thus, from Theorem 3.1(i) we obtain $P_{p-1}(\frac{r}{m},0)=P_{p-1}(-1-\frac{r}{m},0)=P_{p-1}(2n+1+pt,0)\equiv 0 \pmod{p^2}$. If $r\equiv -sp \pmod{2m}$ for some $s\in\{1,2,\ldots,m-1\}$, setting $n=\frac{sp+r}{2m}$ and $t=-\frac{s}{m}$ we find that $n\in\{0,1,\ldots,\frac{p-1}{2}\}$ and $2n+pt=\frac{r}{m}$. Now applying Theorem 3.1(ii) we deduce the result. The proof is now complete. □

From Corollary 2.2 or Theorem 3.2 we deduce the following result.

**Theorem 3.3.** *Let $p$ be an odd prime. Then*

$$\begin{aligned}
&P_{p-1}(1/2,0)\equiv 0 \pmod{p^2} \quad \textit{for}\quad p\equiv 1 \pmod 4,\\
&P_{p-1}(1/3,0)\equiv 0 \pmod{p^2} \quad \textit{for}\quad p\equiv 1 \pmod 3,\\
&P_{p-1}(1/4,0)\equiv 0 \pmod{p^2} \quad \textit{for}\quad p\equiv 1,3 \pmod 8,\\
&P_{p-1}(1/5,0)\equiv 0 \pmod{p^2} \quad \textit{for}\quad p\equiv 1,2 \pmod 5,\\
&P_{p-1}(1/6,0)\equiv 0 \pmod{p^2} \quad \textit{for}\quad p\equiv 1 \pmod 4,\\
&P_{p-1}(1/7,0)\equiv 0 \pmod{p^2} \quad \textit{for}\quad p\equiv 1,3,5 \pmod 7,\\
&P_{p-1}(1/8,0)\equiv 0 \pmod{p^2} \quad \textit{for}\quad p\equiv 1,7,11,13 \pmod{16},\\
&P_{p-1}(1/9,0)\equiv 0 \pmod{p^2} \quad \textit{for}\quad p\equiv 1,2,4 \pmod 9,\\
&P_{p-1}(1/10,0)\equiv 0 \pmod{p^2} \quad \textit{for}\quad p\equiv 1,3,7,9 \pmod{20},\\
&P_{p-1}(1/11,0)\equiv 0 \pmod{p^2} \quad \textit{for}\quad p\equiv 1,4,5,8,9 \pmod{11},\\
&P_{p-1}(1/12,0)\equiv 0 \pmod{p^2} \quad \textit{for}\quad p\equiv 1,5,7,11 \pmod{24}.
\end{aligned}$$

**Lemma 3.1 ([L]).** *Let $p$ be an odd prime. Then*

(i) $H_{\frac{p-1}{2}} \equiv -2q_p(2) \pmod p$, $H_{[\frac p4]} \equiv -3q_p(2) \pmod p$.

(ii) *For $p>3$ we have* $H_{[\frac p3]} \equiv -\frac32 q_p(3) \pmod p$ *and* $H_{[\frac p6]} \equiv -2q_p(2)-\frac32 q_p(3) \pmod p$.

**Theorem 3.4.** *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{54^k} \equiv \begin{cases} 2A-\frac{p}{2A} \pmod{p^2} & \text{if } 3\mid p-1,\ p=A^2+3B^2 \text{ and } 3\mid A-1,\\ 0 \pmod{p^2} & \text{if } 3\mid p-2.\end{cases}$$

Proof. From Theorem 3.2(i) we see that

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{54^k} &= P_{p-1}\Big(-\frac13,0\Big)\\ &\equiv \begin{cases} \binom{\frac{p-1}{2}}{\frac{p-1}{3}}(1+p(\frac13 H_{\frac{2(p-1)}{3}}+\frac16 H_{\frac{p-1}{3}}+\frac23 q_p(2))) \pmod{p^2} & \text{if } p\equiv 1 \pmod 3,\\ 0 \pmod{p^2} & \text{if } p\equiv 2 \pmod 3.\end{cases}\end{aligned}$$

Now we assume $p\equiv 1 \pmod 3$ and $p=A^2+3B^2$ with $A,B\in\mathbb{Z}$ and $A\equiv 1 \pmod 3$. From Lemma 3.1 we have $H_{\frac{p-1}{3}} \equiv -\frac32 q_p(3) \pmod p$ and

$$H_{\frac{2(p-1)}{3}} = H_{p-1}-\sum_{k=1}^{(p-1)/3}\frac{1}{p-k} \equiv \sum_{k=1}^{(p-1)/3}\frac1k \equiv -\frac32 q_p(3) \pmod p.$$

Thus,

$$\begin{aligned}\frac13 H_{\frac{2(p-1)}{3}}+\frac16 H_{\frac{p-1}{3}}+\frac23 q_p(2) &\equiv -\frac12 q_p(3)-\frac14 q_p(3)+\frac23 q_p(2)\\ &= \frac23 q_p(2)-\frac34 q_p(3) \pmod p.\end{aligned}$$

By [BEW, Theorem 9.4.4],

$$\binom{\frac{p-1}{2}}{\frac{p-1}{3}} = \binom{\frac{p-1}{2}}{\frac{p-1}{6}} \equiv \Big(2A-\frac{p}{2A}\Big)\Big(1-\frac23 q_p(2)p+\frac34 q_p(3)p\Big) \pmod{p^2}.$$

Therefore,

$$\begin{aligned}P_{p-1}\Big(-\frac13,0\Big) &\equiv \binom{\frac{p-1}{2}}{\frac{p-1}{3}}\Big(1+p(\frac13 H_{\frac{2(p-1)}{3}}+\frac16 H_{\frac{p-1}{3}}+\frac23 q_p(2))\Big)\\ &\equiv \Big(2A-\frac{p}{2A}\Big)\Big(1-p\Big(\frac23 q_p(2)-\frac34 q_p(3)\Big)\Big)\Big(1+p\Big(\frac23 q_p(2)-\frac34 q_p(3)\Big)\Big)\\ &\equiv 2A-\frac{p}{2A} \pmod{p^2}.\end{aligned}$$

This completes the proof. □

**Remark 3.1** In [S2] the author conjectured Theorem 3.4 and proved the congruence modulo $p$. In [Su4], Z.W. Sun proved the result for $p\equiv 2 \pmod 3$.

## 4. Congruences for $\sum_{k=0}^{p-1}\binom{a}{k}\binom{b-a}{k} \pmod{p^2}$.

Let $n$ be a nonnegative integer. For two variables $a$ and $b$ we define

$$S_n(a,b)=\sum_{k=0}^{n}\binom{a}{k}\binom{b-a}{k}. \tag{4.1}$$

Set

$$F(a,k)=\binom{a}{k}\binom{b-a}{k}\quad\text{and}\quad G(a,k)=(2a-b+1)\binom{a}{k-1}\binom{b-a-1}{k-1}.$$

It is easy to see that

$$(a-b)F(a,k)+(a+1)F(a+1,k)=G(a,k+1)-G(a,k).$$

Thus,

$$\begin{aligned}&(a-b)\sum_{k=0}^{n}F(a,k)+(a+1)\sum_{k=0}^{n}F(a+1,k)\\&=\sum_{k=0}^{n}(G(a,k+1)-G(a,k))=G(a,n+1)-G(a,0)=G(a,n+1).\end{aligned}$$

That is,

$$(a-b)S_n(a,b)+(a+1)S_n(a+1,b)=(2a-b+1)\binom{a}{n}\binom{b-a-1}{n}. \tag{4.2}$$

**Lemma 4.1.** *Let $p$ be an odd prime and $b,c,t\in\mathbb{Z}_p$. Then*

$$\sum_{k=0}^{p-1}\binom{pt}{k}\binom{b+cpt}{k}\equiv 1+pt\sum_{k=1}^{\langle b\rangle_p}\frac{1}{k} \pmod{p^2}.$$

Proof. Clearly

$$\begin{aligned}&\sum_{k=0}^{p-1}\binom{pt}{k}\binom{b+cpt}{k}\\
&=1+\sum_{k=1}^{p-1}\frac{pt}{k}\binom{pt-1}{k-1}\binom{b+cpt}{k}\equiv 1+\sum_{k=1}^{p-1}\frac{pt}{k}\binom{-1}{k-1}\binom{b}{k}\\
&\equiv 1+\sum_{k=1}^{p-1}\frac{pt}{k}(-1)^{k-1}\binom{\langle b\rangle_p}{k}=1+pt\sum_{k=1}^{\langle b\rangle_p}\frac{(-1)^{k-1}}{k}\binom{\langle b\rangle_p}{k}\\
&=1+pt\sum_{k=1}^{\langle b\rangle_p}(-1)^{k-1}\binom{\langle b\rangle_p}{k}\int_0^1 x^{k-1}\,dx=1+pt\int_0^1\sum_{k=1}^{\langle b\rangle_p}\binom{\langle b\rangle_p}{k}(-x)^{k-1}\,dx\\
&=1+pt\int_0^1\frac{(1-x)^{\langle b\rangle_p}-1}{-x}\,dx=1+pt\int_0^1\frac{u^{\langle b\rangle_p}-1}{u-1}du\\
&=1+pt\int_0^1\sum_{k=0}^{\langle b\rangle_p-1}u^k\,du=1+pt\sum_{k=0}^{\langle b\rangle_p-1}\frac{1}{k+1} \pmod{p^2}.\end{aligned}$$

This proves the lemma. □

**Lemma 4.2.** *Let $p$ be an odd prime, $m \in \{1, 2, \dots, p-1\}$ and $t \in \mathbb{Z}_p$. Then*

$$\binom{m+pt-1}{p-1} \equiv \frac{pt}{m} - \frac{p^2t^2}{m^2} + \frac{p^2t}{m}\sum_{k=1}^{m}\frac{1}{k} \pmod{p^3}.$$

Proof. For $m < \frac{p}{2}$ we see that

$$\begin{aligned}\binom{m+pt-1}{p-1} &= \frac{(m-1+pt)(m-2+pt)\cdots(1+pt)\cdot pt(pt-1)\cdots(pt-(p-1-m))}{(p-1)!} \\ &= \frac{pt(p^2t^2-1^2)\cdots(p^2t^2-(m-1)^2)(pt-m)\cdots(pt-(p-1-m))}{(p-1)!} \\ &\equiv pt\frac{(m-1)!(-1)(-2)\cdot\cdots\cdot(-(p-1-m))}{(p-1)!}\Big(1-pt\sum_{k=m}^{p-1-m}\frac{1}{k}\Big) \\ &= pt\cdot\frac{(-1)^{p-1-m}\cdot(m-1)!}{(p-m)\cdots(p-1)}\Big(1-pt\big(H_{p-1-m}-H_m+\frac{1}{m}\big)\Big) \pmod{p^3}.\end{aligned}$$

For $m > \frac{p}{2}$ we also have

$$\begin{aligned}\binom{m+pt-1}{p-1} &= \frac{(m-1+pt)(m-2+pt)\cdots(1+pt)\cdot pt(pt-1)\cdots(pt-(p-1-m))}{(p-1)!} \\ &= \frac{pt(p^2t^2-1^2)\cdots(p^2t^2-(p-1-m)^2)(pt+p-m)\cdots(pt+m-1)}{(p-1)!} \\ &\equiv pt\frac{(-1)^{p-1-m}(p-1-m)!(m-1)!}{(p-1)!}\Big(1+pt\sum_{k=p-m}^{m-1}\frac{1}{k}\Big) \\ &= pt\cdot\frac{(-1)^{p-1-m}\cdot(m-1)!}{(p-m)\cdots(p-1)}\Big(1-pt\big(H_{p-1-m}-H_m+\frac{1}{m}\big)\Big) \pmod{p^3}.\end{aligned}$$

Since $(p-m)\cdots(p-1) \equiv (-1)^m m!(1-pH_m) \pmod{p^2}$, by the above we get

$$\begin{aligned}\binom{m+pt-1}{p-1} &\equiv pt\cdot\frac{(-1)^m\cdot(m-1)!}{(-1)^m\cdot m!(1-pH_m)}\Big(1-\frac{pt}{m}-pt(H_{p-1-m}-H_m)\Big) \\ &\equiv \frac{pt}{m}(1+pH_m)\Big(1-\frac{pt}{m}-pt(H_{p-1-m}-H_m)\Big) \\ &\equiv \frac{pt}{m}-\frac{p^2t}{m}\Big(\frac{t}{m}+t(H_{p-1-m}-H_m)-H_m\Big) \pmod{p^3}.\end{aligned}$$

To see the result, we note that

$$\begin{aligned}H_{p-1-m}-H_m &= \sum_{r=1}^{p-1-m}\frac{1}{r}-\sum_{k=1}^{m}\frac{1}{k} = \sum_{k=m+1}^{p-1}\frac{1}{p-k}-\sum_{k=1}^{m}\frac{1}{k} \\ &\equiv -\sum_{k=m+1}^{p-1}\frac{1}{k}-\sum_{k=1}^{m}\frac{1}{k} = -\sum_{r=1}^{(p-1)/2}\Big(\frac{1}{r}+\frac{1}{p-r}\Big) \\ &\equiv 0 \pmod{p}. \quad \square\end{aligned}$$

**Lemma 4.3.** *Let $p$ be an odd prime, $m \in \{1, 2, \dots, p-1\}$ and $b, t \in \mathbb{Z}_p$. Then*

$$\binom{b-(m+pt)}{p-1} \equiv \begin{cases} 1 \pmod{p^2} & \text{if } m = \langle b\rangle_p + 1, \\ \frac{b-\langle b\rangle_p - pt}{b+1-m} \pmod{p^2} & \text{if } m \le \langle b\rangle_p, \\ \frac{b-\langle b\rangle_p - p(t+1)}{b+1-m} \pmod{p^2} & \text{if } m > \langle b\rangle_p + 1. \end{cases}$$

Proof. If $m = \langle b\rangle_p + 1$, setting $b - m = rp + p - 1$ we find $r \in \mathbb{Z}_p$ and so

$$\binom{b-m-pt}{p-1} = \frac{(p-1+p(r-t))(p-2+p(r-t))\cdots(1+p(r-t))}{(p-1)!}$$
$$\equiv 1 + p(r-t)\sum_{k=1}^{p-1}\frac{1}{k} = 1 + p(r-t)\sum_{k=1}^{\frac{p-1}{2}}\Big(\frac{1}{k} + \frac{1}{p-k}\Big) \equiv 1 \pmod{p^2}.$$

Now we assume $m \ne \langle b\rangle_p + 1$ and $b - m = \langle b-m\rangle_p + pr$. Then $r \in \mathbb{Z}_p$, $\langle b-m\rangle_p \le p-2$ and

$$\binom{b-(m+pt)}{p-1} = \frac{1}{(p-1)!}\prod_{k=0}^{p-2}(\langle b-m\rangle_p - k + p(r-t)) \equiv \frac{p(r-t)}{(p-1)!}\prod_{\substack{k=0\\ k\ne\langle b-m\rangle_p}}^{p-2}(\langle b-m\rangle_p - k)$$
$$\equiv \frac{p(r-t)}{\langle b-m\rangle_p - p + 1} \equiv \frac{p(r-t)}{b-m+1} = \frac{b-m-\langle b-m\rangle_p - pt}{b-m+1} \pmod{p^2}.$$

Since

$$\langle b-m\rangle_p = \begin{cases} \langle b\rangle_p - m & \text{if } m \le \langle b\rangle_p, \\ p + \langle b\rangle_p - m & \text{if } m > \langle b\rangle_p + 1, \end{cases}$$

we see that

$$b - m - \langle b-m\rangle_p = \begin{cases} b - m - (\langle b\rangle_p - m) = b - \langle b\rangle_p & \text{if } m \le \langle b\rangle_p, \\ b - m - (p + \langle b\rangle_p - m) = b - \langle b\rangle_p - p & \text{if } m > \langle b\rangle_p + 1. \end{cases}$$

Now combining all the above we obtain the result. □

**Theorem 4.1.** *Let $p$ be an odd prime and $a, b \in \mathbb{Z}_p$. Then*

$$\sum_{k=0}^{p-1}\binom{a}{k}\binom{b-a}{k} \equiv \begin{cases} \frac{(-1)^{\langle a\rangle_p - \langle b\rangle_p - 1}}{(\langle a\rangle_p - \langle b\rangle_p)\binom{\langle a\rangle_p}{\langle b\rangle_p}}(b - \langle b\rangle_p) \pmod{p^2} & \text{if } \langle a\rangle_p > \langle b\rangle_p, \\ \binom{\langle b\rangle_p}{\langle a\rangle_p}(1 + (b - \langle b\rangle_p)H_{\langle b\rangle_p} - (a - \langle a\rangle_p)H_{\langle a\rangle_p} & \\ \quad -(b - a - \langle b-a\rangle_p)H_{\langle b-a\rangle_p}) \pmod{p^2} & \text{if } \langle a\rangle_p \le \langle b\rangle_p. \end{cases}$$

Proof. If $\langle a\rangle_p = 0$, the result follows from Lemma 4.1 (with $pt = a$ and $c = -1$). From now on we assume $\langle a\rangle_p \ge 1$. Set $a = \langle a\rangle_p + pt$ and $b = \langle b\rangle_p + ps$. Then $s, t \in \mathbb{Z}_p$. For

$m\in\{1,2,\dots,p-1\}$, by (4.2) and Lemmas 4.2-4.3 we obtain

$$\begin{aligned}&(m+pt)S_{p-1}(m+pt,b)+(m+pt-b-1)S_{p-1}(m+pt-1,b)\\&=(2(m+pt)-1-b)\binom{m+pt-1}{p-1}\binom{b-(m+pt)}{p-1}\\&\equiv\begin{cases}(2m-1-b)\frac{pt}{m}+p^2t(\frac{t}{m}+H_m)\pmod{p^3} & \text{if } m=\langle b\rangle_p+1,\\(2m-1-b)\frac{pt}{m}\cdot\frac{b-\langle b\rangle_p-pt}{b+1-m}\pmod{p^3} & \text{if } m\le\langle b\rangle_p,\\(2m-1-b)\frac{pt}{m}\cdot\frac{b-\langle b\rangle_p-p(t+1)}{b+1-m}\pmod{p^3} & \text{if } m>\langle b\rangle_p+1.\end{cases}\end{aligned}\tag{4.3}$$

Hence, if $1\le\langle a\rangle_p\le\langle b\rangle_p$, then

$$\begin{aligned}&S_{p-1}(a,b)\\&=S_{p-1}(\langle a\rangle_p+pt,b)\equiv-\frac{\langle a\rangle_p+pt-b-1}{\langle a\rangle_p+pt}S_{p-1}(\langle a\rangle_p-1+pt,b)\\&=\frac{1+\langle b\rangle_p-\langle a\rangle_p+p(s-t)}{\langle a\rangle_p+pt}S_{p-1}(\langle a\rangle_p-1+pt,b)\\&\equiv\frac{1+\langle b\rangle_p-\langle a\rangle_p+p(s-t)}{\langle a\rangle_p+pt}\cdot\frac{1+\langle b\rangle_p-(\langle a\rangle_p-1)+p(s-t)}{\langle a\rangle_p-1+pt}S_{p-1}(\langle a\rangle_p-2+pt,b)\\&\equiv\cdots\equiv\prod_{k=1}^{\langle a\rangle_p}\frac{1+\langle b\rangle_p-k+p(s-t)}{k+pt}\cdot S_{p-1}(pt,b)\pmod{p^2}.\end{aligned}$$

Now applying Lemma 4.1 we see that for $1\le\langle a\rangle_p\le\langle b\rangle_p$,

$$\begin{aligned}&S_{p-1}(a,b)\\&\equiv\frac{\langle b\rangle_p(\langle b\rangle_p-1)\cdots(\langle b\rangle_p-\langle a\rangle_p+1)(1+p(s-t)(H_{\langle b\rangle_p}-H_{\langle b-a\rangle_p}))}{\langle a\rangle_p!(1+ptH_{\langle a\rangle_p})}(1+ptH_{\langle b\rangle_p})\\&\equiv\binom{\langle b\rangle_p}{\langle a\rangle_p}(1+p(s-t)(H_{\langle b\rangle_p}-H_{\langle b-a\rangle_p}))(1-ptH_{\langle a\rangle_p})(1+ptH_{\langle b\rangle_p})\\&\equiv\binom{\langle b\rangle_p}{\langle a\rangle_p}(1+psH_{\langle b\rangle_p}-ptH_{\langle a\rangle_p}-(ps-pt)H_{\langle b-a\rangle_p})\\&=\binom{\langle b\rangle_p}{\langle a\rangle_p}(1+(b-\langle b\rangle_p)H_{\langle b\rangle_p}-(a-\langle a\rangle_p)H_{\langle a\rangle_p}-(b-a-\langle b-a\rangle_p)H_{\langle b-a\rangle_p})\pmod{p^2}.\end{aligned}$$

Now we assume $\langle a\rangle_p>\langle b\rangle_p$. Clearly

$$S_{p-1}(\langle b\rangle_p+pt,b)=1+\sum_{k=1}^{p-1}\binom{\langle b\rangle_p+pt}{k}\frac{b-\langle b\rangle_p-pt}{k}\binom{b-\langle b\rangle_p-pt-1}{k-1}\equiv1\pmod{p}.$$

This together with (4.3) yields

$$\begin{aligned}&(\langle b\rangle_p+1+pt)S_{p-1}(\langle b\rangle_p+1+pt,b)\\&\equiv-(\langle b\rangle_p+pt-b)S_{p-1}(\langle b\rangle_p+pt,b)+(2(\langle b\rangle_p+1)-1-b)\frac{pt}{\langle b\rangle_p+1}\\&\equiv b-\langle b\rangle_p-pt+pt=b-\langle b\rangle_p\pmod{p^2}.\end{aligned}$$

Thus,

$$S_{p-1}(\langle b\rangle_p+1+pt,b)\equiv\frac{b-\langle b\rangle_p}{\langle b\rangle_p+1+pt}\equiv\frac{b-\langle b\rangle_p}{\langle b\rangle_p+1}\pmod{p^2}. \tag{4.4}$$

This shows that the result is true in the case $\langle a\rangle_p=\langle b\rangle_p+1$. Now from (4.3)-(4.4) we deduce that for $\langle a\rangle_p>\langle b\rangle_p+1$,

$$\begin{aligned}S_{p-1}(a,b)&=S_{p-1}(\langle a\rangle_p+pt,b)\equiv\Big(\frac{b+1}{\langle a\rangle_p+pt}-1\Big)S_{p-1}(\langle a\rangle_p+pt-1,b)\\&\equiv\Big(\frac{b+1}{\langle a\rangle_p+pt}-1\Big)\Big(\frac{b+1}{\langle a\rangle_p-1+pt}-1\Big)S_{p-1}(\langle a\rangle_p-2+pt,b)\\&\equiv\cdots\equiv\prod_{k=\langle b\rangle_p+2}^{\langle a\rangle_p}\Big(\frac{b+1}{k+pt}-1\Big)\cdot S_{p-1}(\langle b\rangle_p+1+pt,b)\\&\equiv\prod_{k=\langle b\rangle_p+2}^{\langle a\rangle_p}\frac{\langle b\rangle_p+1-k}{k}\cdot\frac{b-\langle b\rangle_p}{\langle b\rangle_p+1}\\&=\frac{(-1)^{\langle a\rangle_p-\langle b\rangle_p-1}}{(\langle a\rangle_p-\langle b\rangle_p)\binom{\langle a\rangle_p}{\langle b\rangle_p}}(b-\langle b\rangle_p)\pmod{p^2}.\end{aligned}$$

This completes the proof. □

**Remark 4.1** Let $p$ be an odd prime and $a\in\mathbb{Z}_p$. Taking $b=-1$ in Theorem 4.1 and then applying the fact $H_{\langle -1-a\rangle_p}=H_{\langle p-1-a\rangle_p}\equiv H_{\langle a\rangle_p}\pmod p$ we deduce Corollary 2.1. In [Su2], Z.W. Sun showed that

$$\sum_{k=0}^{p-1}\binom{a}{k}^2\equiv\binom{2a}{\langle a\rangle_p}\pmod{p^2}.$$

This can be deduced from Theorem 4.1 by taking $b=2a$.

**Corollary 4.1.** *Let $p$ be an odd prime and $a,b\in\mathbb{Z}_p$. Then*

$$\sum_{k=0}^{p-1}\binom{a}{k}\binom{b-a}{k}\equiv\binom{\langle b\rangle_p}{\langle a\rangle_p}\pmod p.$$

**Corollary 4.2.** *Let $p$ be an odd prime and $a\in\mathbb{Z}_p$. Then*

$$\sum_{k=0}^{p-1}\binom{a}{k}\binom{-a}{k}\equiv\begin{cases}0\pmod{p^2} & \text{if } a\not\equiv 0\pmod p,\\ 1\pmod{p^2} & \text{if } a\equiv 0\pmod p.\end{cases}$$

Proof. Taking $b=0$ in Theorem 4.1 we deduce the result. □

**Corollary 4.3.** *Let $p$ be an odd prime and $a \in \mathbb{Z}_p$. Then*

$$\sum_{k=0}^{p-1}\binom{a}{k}\binom{1-a}{k} \equiv \begin{cases} 0 \pmod{p^2} & \text{if } a(1-a) \not\equiv 0 \pmod p, \\ 1+a \pmod{p^2} & \text{if } a \equiv 0 \pmod p, \\ 2-a \pmod{p^2} & \text{if } a \equiv 1 \pmod p. \end{cases}$$

Proof. Taking $b = 1$ in Theorem 4.1 we deduce the result. □

Using the method in the proof of Lemma 2.4 we can show that for any positive integer $n$,

$$\sum_{k=0}^{n}\binom{a}{k}\binom{-a}{k} = \binom{n+a}{n}\binom{n-a}{n}, \tag{4.5}$$

$$\sum_{k=0}^{n}\binom{a}{k}\binom{1-a}{k} = -\frac{a^2-a-n}{n^2}\binom{a-2}{n-1}\binom{-a-1}{n-1}. \tag{4.6}$$

**Theorem 4.2.** *Let $p$ be an odd prime. Then*

$$\sum_{k=0}^{p-1}\binom{-\frac14}{k}\binom{-\frac12}{k} \equiv \begin{cases} (-1)^{\frac{p+1}{4}}p\binom{\frac{p-1}{2}}{\frac{p+1}{4}}^{-1} \pmod{p^2} & \text{if } p \equiv 3 \pmod 4, \\ (-1)^{\frac{p-1}{4}}(2x-\frac{p}{2x}) \pmod{p^2} & \text{if } p = x^2+y^2 \equiv 1 \pmod 4 \text{ and } 4 \mid x-1. \end{cases}$$

Proof. Set $a = -\frac14$ and $b = -\frac34$. Then

$$\langle a\rangle_p = \begin{cases} \frac{p-1}{4} & \text{if } p \equiv 1 \pmod 4, \\ \frac{3p-1}{4} & \text{if } p \equiv 3 \pmod 4, \end{cases} \quad \text{and} \quad \langle b\rangle_p = \begin{cases} \frac{3p-3}{4} & \text{if } p \equiv 1 \pmod 4, \\ \frac{p-3}{4} & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

If $p \equiv 3 \pmod 4$, then $\langle a\rangle_p > \langle b\rangle_p$. Thus, by Theorem 4.1 we have

$$\sum_{k=0}^{p-1}\binom{-\frac14}{k}\binom{-\frac12}{k} \equiv \frac{(-1)^{\frac{3p-1}{4}-\frac{p-3}{4}-1}}{(\frac{3p-1}{4}-\frac{p-3}{4})\binom{\frac{3p-1}{4}}{\frac{p-3}{4}}}\Big(-\frac34-\frac{p-3}{4}\Big) \equiv \frac{3p}{2}\binom{\frac{3(p+1)}{4}}{\frac{p+1}{4}}^{-1} \pmod{p^2}.$$

Since

$$\begin{aligned}\binom{\frac{3(p+1)}{4}}{\frac{p+1}{4}} &= \frac{(p-\frac{p-3}{4})(p-\frac{p+1}{4})\cdots(p-\frac{p-3}{2})}{\frac{p+1}{4}!} \\ &\equiv (-1)^{\frac{p+1}{4}}\frac{\frac{p-3}{2}\cdots\frac{p-3}{4}}{\frac{p+1}{4}!} = (-1)^{\frac{p+1}{4}}\binom{\frac{p-1}{2}}{\frac{p+1}{4}}\cdot\frac{(p-3)/4}{(p-1)/2} \\ &\equiv \frac32(-1)^{\frac{p+1}{4}}\binom{\frac{p-1}{2}}{\frac{p+1}{4}} \pmod p,\end{aligned}$$

by the above we obtain the result in the case $p \equiv 3 \pmod 4$.

Now we assume $p\equiv 1 \pmod 4$ and so $p=x^2+y^2$ with $x,y\in\mathbb{Z}$ and $x\equiv 1 \pmod 4$. By the proof of Lemma 4.2 and Lemma 3.1(i) we have $H_{\frac{3(p-1)}{4}}\equiv H_{\frac{p-1}{4}}\equiv -3q_p(2) \pmod p$ and $H_{\frac{p-1}{2}}\equiv -2q_p(2) \pmod p$. Now applying the above and Theorem 4.1 we deduce that

$$\begin{aligned}\sum_{k=0}^{p-1}\binom{-\frac14}{k}\binom{-\frac12}{k} &\equiv \binom{\frac{3(p-1)}{4}}{\frac{p-1}{4}}\Big(1-\frac{3p}{4}H_{\frac{3(p-1)}{4}}+\frac{p}{4}H_{\frac{p-1}{4}}+\frac{p}{2}H_{\frac{p-1}{2}}\Big)\\ &\equiv \binom{\frac{3(p-1)}{4}}{\frac{p-1}{4}}\Big(1-\frac{p}{2}(-3q_p(2))+\frac{p}{2}(-2q_p(2))\Big)\\ &= \binom{\frac{3(p-1)}{4}}{\frac{p-1}{4}}\Big(1+\frac12 pq_p(2)\Big) \pmod{p^2}.\end{aligned}$$

By [BEW, Theorem 9.4.3] we have

$$\binom{\frac{3(p-1)}{4}}{\frac{p-1}{4}}\equiv\Big(2x-\frac{p}{2x}\Big)(-1)^{\frac{p-1}{4}}\Big(1-\frac12 pq_p(2)\Big) \pmod{p^2}.$$

Hence

$$\sum_{k=0}^{p-1}\binom{-\frac14}{k}\binom{-\frac12}{k}\equiv(-1)^{\frac{p-1}{4}}\Big(2x-\frac{p}{2x}\Big) \pmod{p^2}.$$

This proves the result in the case $p\equiv 1 \pmod 4$. The proof is now complete. □

**Theorem 4.3.** *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\binom{-\frac16}{k}\binom{-\frac13}{k}\equiv\begin{cases}\frac{3p}{2}\binom{\frac{p-1}{2}}{\frac{p-5}{6}}^{-1} \pmod{p^2} & \text{if } p\equiv 2 \pmod 3,\\ 2A-\frac{p}{2A} \pmod{p^2} & \text{if } p=A^2+3B^2\equiv 1 \pmod 3 \text{ and } 3\mid A-1.\end{cases}$$

Proof. Set $a=-\frac16$ and $b=-\frac12$. Then

$$\langle b\rangle_p=\frac{p-1}{2}\quad\text{and}\quad\langle a\rangle_p=\begin{cases}\frac{p-1}{6} & \text{if } p\equiv 1 \pmod 3,\\ \frac{5p-1}{6} & \text{if } p\equiv 2 \pmod 3.\end{cases}$$

For $p\equiv 2 \pmod 3$ we have $\langle a\rangle_p>\langle b\rangle_p$. Thus, by Theorem 4.1 we get

$$\sum_{k=0}^{p-1}\binom{-\frac16}{k}\binom{-\frac13}{k}\equiv\frac{(-1)^{\frac{5p-1}{6}-\frac{p-1}{2}-1}}{(\frac{5p-1}{6}-\frac{p-1}{2})\binom{\frac{5p-1}{6}}{\frac{p-1}{2}}}\Big(-\frac12-\frac{p-1}{2}\Big)=\frac{3p}{2(p+1)\binom{\frac{5p-1}{6}}{\frac{p-1}{2}}} \pmod{p^2}.$$

Note that

$$\begin{aligned}\binom{\frac{5p-1}{6}}{\frac{p-1}{2}}&=\binom{\frac{5p-1}{6}}{\frac{p+1}{3}}=\binom{p-\frac{p+1}{6}}{\frac{p+1}{3}}\equiv\binom{-\frac{p+1}{6}}{\frac{p+1}{3}}\\ &=\binom{\frac{p+1}{6}+\frac{p+1}{3}-1}{\frac{p+1}{3}}=\binom{\frac{p-1}{2}}{\frac{p+1}{3}}=\binom{\frac{p-1}{2}}{\frac{p-5}{6}} \pmod p.\end{aligned}$$

We then get the result in the case $p\equiv 2\pmod 3$.

Now we assume $p\equiv 1\pmod 3$ and so $p=A^2+3B^2\equiv 1\pmod 3$ with $3\mid A-1$. By Theorem 4.1 and Lemma 3.1 we obtain

$$\begin{aligned}\sum_{k=0}^{p-1}\binom{-\frac16}{k}\binom{-\frac13}{k}&\equiv\binom{\frac{p-1}{2}}{\frac{p-1}{6}}\Big(1-\frac p2H_{\frac{p-1}{2}}+\frac p6H_{\frac{p-1}{6}}+\frac p3H_{\frac{p-1}{3}}\Big)\\&\equiv\binom{\frac{p-1}{2}}{\frac{p-1}{6}}\Big(1-\frac p2(-2q_p(2))+\frac p6\big(-2q_p(2)-\frac32q_p(3)\big)+\frac p3\big(-\frac32q_p(3)\big)\Big)\\&=\binom{\frac{p-1}{2}}{\frac{p-1}{6}}\Big(1+p\big(\frac23q_p(2)-\frac34q_p(3)\big)\Big)\pmod{p^2}.\end{aligned}$$

By [BEW, Theorem 9.4.4],

$$\binom{\frac{p-1}{2}}{\frac{p-1}{6}}\equiv\Big(2A-\frac{p}{2A}\Big)\Big(1-p\big(\frac23q_p(2)-\frac34q_p(3)\big)\Big)\pmod{p^2}.$$

Therefore,

$$\sum_{k=0}^{p-1}\binom{-\frac16}{k}\binom{-\frac13}{k}\equiv\Big(2A-\frac{p}{2A}\Big)\Big(1^2-p^2\big(\frac23q_p(2)-\frac34q_p(3)\big)^2\Big)\equiv 2A-\frac{p}{2A}\pmod{p^2}.$$

This proves the result in the case $p\equiv 1\pmod 3$. Hence the theorem is proved. □

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